Hyperplane Arrangements and Matroids for Linear Complementarity Problems

Computing the B-differential of the Componentwise Minimum of Affine Functions

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Outline

Plan

Systems of smooth nonlinear equations

General problem:

Find a point $x_* \in \mathbb{R}^n$: $F(x_*) = 0$, with $F : \mathbb{R}^n \to \mathbb{R}^n$ smooth

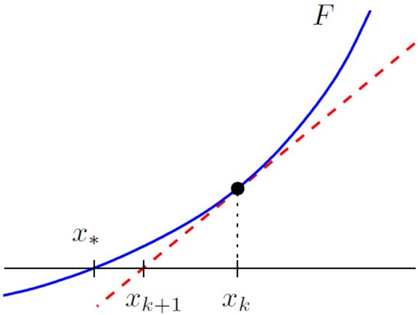


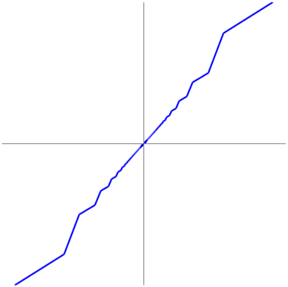
Illustration of Newton's method in 1D

x_0 near x_*,
$F \in \mathcal{C}^{1,1}$,
$F'(x_*)$ non-singular
$\Downarrow$
quadratic convergence

Nonsmooth equations

Harder problem:

Find a poind $x_* \in \mathbb{R}^n$: $F(x_*) = 0$, with $F : \mathbb{R}^n \to \mathbb{R}^n$ nonsmooth



Kummer's counter-example to Newton; in the nonsmooth case the Jacobian might not be defined.

x_0 near x_*,
F semismooth,
all $J \in$ "$F'(x_*)$"
non-singular
$\Downarrow$
quadratic
convergence

Remedy: semismooth Newton's method

Adaptation of the usual method for this difficulty ([Qi93; QS93])
Replaces $F'(x_k)$ with a "generalized Jacobian" J_k

Algorithm's sketch

- take $x_0 \in \mathbb{R}^n$ (near x_*)
- for k = 1 2, ..., solve $F(x_k) + J_k\delta_k = 0$ for δ_k, with $J_k \in \partial_B F(x_k)$: $\partial_B F$ is the Bouligand differential
- then $x_{k+1} = x_k + (\alpha_k)\delta_k$

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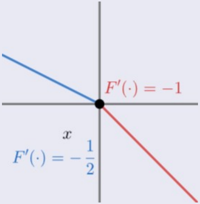
Generalized derivatives

Bouligand differential

$$\partial_{\mathrm{B}} F(x) = \{J \in \mathbb{R}^{n\times n} : \exists\ (x_k)_k \to x, F'(x_k) \to J\} \tag{1}$$

Example: $F(x) = \begin{cases} -x/2 & \text{if } x \leq 0 \\ -x & \text{if } x > 0 \end{cases}$,
$\partial_B F(0) = \{-1/2, -1\}$.



$\underbrace{\partial_B F(x)}_{=??} \subsetneq \underbrace{\partial_B F_1(x) \times \cdots \times \partial_B F_p(x)}_{\prod \cdots = \text{ easy}}$ (sometimes $=$)

$\exists$ other differentials: Clarke, Mordukhovich, 2nd order...

Linear Complementarity Problems

General form [CPS92; FP03]

$$A, B \in \mathbb{R}^{n\times n}, a, b \in \mathbb{R}^n, \to \ \mathcal{A}(x) = Ax + a, \mathcal{B}(x) = Bx + b$$
$$0 \leq (Ax + a) \perp (Bx + b) \geq 0 \Leftrightarrow \tag{2}$$
$$\forall\ i, A_{i,:}x + a_i \geq 0, B_{i,:}x + b_i \geq 0, (A_{i,:}x + a_i)(B_{i,:}x + b_i) = 0$$

Remark: $u \geq 0, v \geq 0, uv = 0 \Leftrightarrow \min(u, v) = 0$

$(2) \Leftrightarrow \forall\ i, F_i(x) := \min(\mathcal{A}_i(x), \mathcal{B}_i(x)) = 0 \Leftrightarrow F(x) = 0$

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Summary

Minimum function on LCPs $\Rightarrow$ semismooth system

Adapted Newton requires info on $\partial_B \min(\mathcal{A}, \mathcal{B})(\cdot) = \partial_B F(\cdot)$

Or to form $\partial_C \min(\mathcal{A}, \mathcal{B})(\cdot)$.

One $J_B \in \partial_B F$: [Qi93]
One $J_C \in \partial_C F$: [CX11]
But all of them?

Upcoming plan

The main question

Determine generalized Jacobians of
$x \mapsto F(x) = \min(Ax + a, Bx + b)$

- their structure
- finite (but exponential) number of elements
- how to compute them

Plan

Computing the B-differential

$f, g \in \mathcal{C}^1$: $\min(f(x), g(x))$ diff $\Leftrightarrow$ $f(x) \neq g(x)$ or $f'(x) = g'(x)$.

Important: F is piecewise affine: F' is piecewise **constant**.

$$\begin{cases} A_{i,:} & = & B_{i,:} \\ A_{i,:}x + a_i & < & B_{i,:}x + b_i \\ A_{i,:}x + a_i & > & B_{i,:}x + b_i \end{cases} \Rightarrow \forall\, J \in \partial_B F(x), J_{i,:} = \begin{cases} A_{i,:} = B_{i,:} \\ A_{i,:} \\ B_{i,:} \end{cases}$$

$I(x) := \{i \in [1:n] : A_{i,:}x + a_i = B_{i,:}x + b_i, A_{i,:} \neq B_{i,:}\}; |I(x)| = p$

$\min(\mathcal{A}, \mathcal{B})$ non-diff $\Leftrightarrow$ affine terms equal $\Leftrightarrow$ **hyperplanes**

$$\mathbb{R}^n = H_i^- \cup H_i \cup H_i^+, \quad H_i^{-,+} = \{x \in \mathbb{R}^n : v_i^\mathsf{T} x <, > 0\}$$

Hyperplanes $H_i := (B_{i,:} - A_{i,:})^\perp := v_i^\perp$; for ∂_B's def, $\mathbb{R}^n \setminus \cup H_i$

H_i^- or H_i^+, $\forall\, i \in [1:p]$: $H_i^+ \Leftrightarrow J_{i,:} = A_{i,:}$, $H_i^- \Leftrightarrow J_{i,:} = B_{i,:}$

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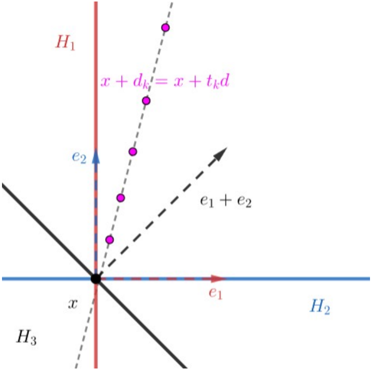
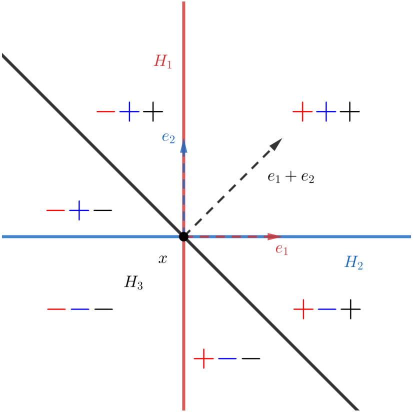
Directions and hyperplanes - 1

$$V = [v_1 \quad v_2 \quad v_3] = [e_1 \quad e_2 \quad e_1 + e_2]$$



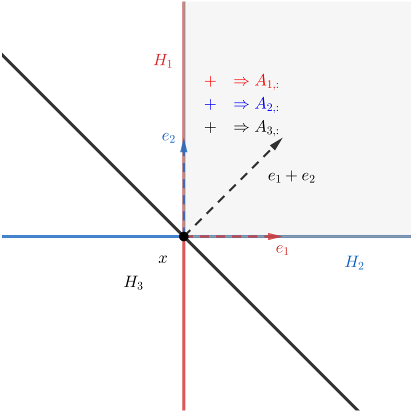
Red, blue, black: H_i and v_i. In magenta, points of the form $x + t_k d$, $t_k \searrow 0$. The points remain on the same sides of the hyperplanes along k: the J is constant; no need for sequences, points are sufficient.

Directions and hyperplanes - 2



$\forall\ i \in [1:p]$, 2 possibilities: maximum of 2^p Jacobians. Here, 6 among the $2^3 = 8$ possible Jacobians exist in $\partial_B F$

Directions and hyperplanes - 3



In the grey area, the points remain on the same sides of the hyperplanes: the Jacobian matrix remains constant.

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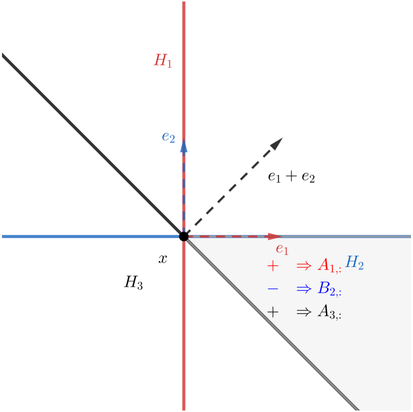

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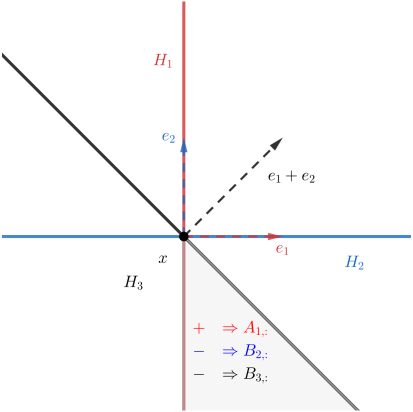

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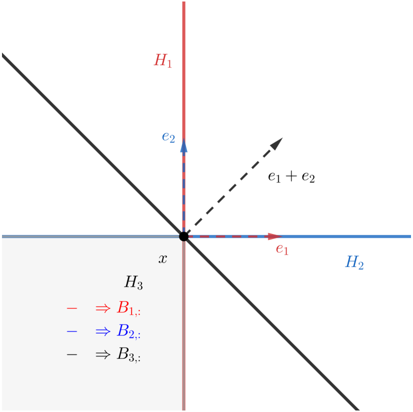

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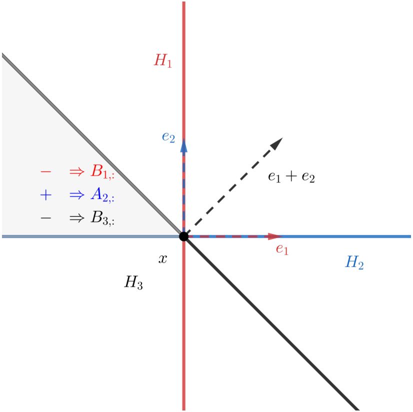

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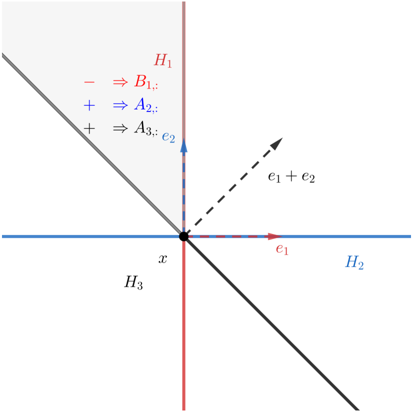

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Summary

$|I(x)| = p$ hyperplanes, $H_i = v_i^{\perp}, v_i = B_{i,:} - A_{i,:}$
$\mathbb{R}^n \setminus \bigcup H_i =$ differentiable points, on the $+$ or $-$ side of every H_i.

Fundamental question

$$\text{given } v_i := (B_{i,:} - A_{i,:})^\top$$
$$\text{find all } s = (s_1, \dots, s_p) \in \{\pm 1\}^p,$$
$$\text{s.t. } \exists\, d_s, \forall\, i \in [1:p], s_i v_i^\top d_s > 0$$

2^p linear feasibility problems to solve... How to improve?

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Very well-known in algebra / combinatorics...
... but very theoretically: Möbius function, lattices, matroids.

Very impressive results such as $|\partial_B F(x)|$ (Winder [Win66])

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\begin{aligned}
|\partial_B F(x)| &= \sum_{T \subset \{H_i, i \in [1:m]\}} (-1)^{|T| - n + \dim(\bigcap H_t, t \in T)} \\
&= \sum_{\mathcal{V} \subset \{v_1, \dots, v_m\}} (-1)^{|\mathcal{V}| - \operatorname{rank}(\mathcal{V})} \\
&= \sum_{\mathcal{V} \subset \{v_1, \dots, v_m\}} (-1)^{\operatorname{null}(\mathcal{V})}
\end{aligned}
$$

or upper bounds but not exactly $\partial_B F(x)$.

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&= \sum_{\mathcal{V} \subset \{v_1, \dots, v_m\}} (-1)^{|\mathcal{V}| - \mathrm{rank}(\mathcal{V})} \\
&= \sum_{\mathcal{V} \subset \{v_1, \dots, v_m\}} (-1)^{\mathrm{null}(\mathcal{V})}
\end{aligned}
$$

or upper bounds but not exactly $\partial_B F(x)$.

Plan

Main reasoning

Algorithm from [RČ18]:

- recursive process that adds hyperplanes one at a time
- uses a tree structure
- each node has one or two descendants,
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Or same tree structure but in a 'dual' way ('dual algorithm').

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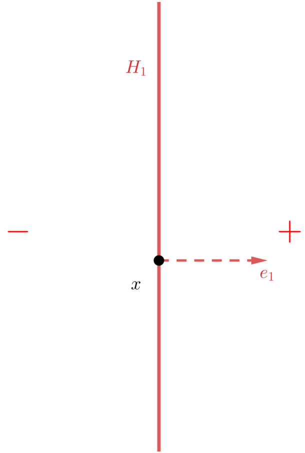

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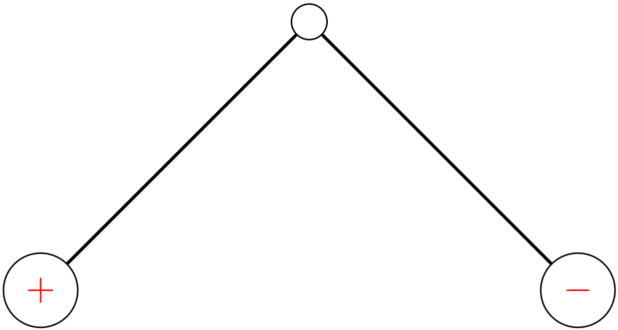

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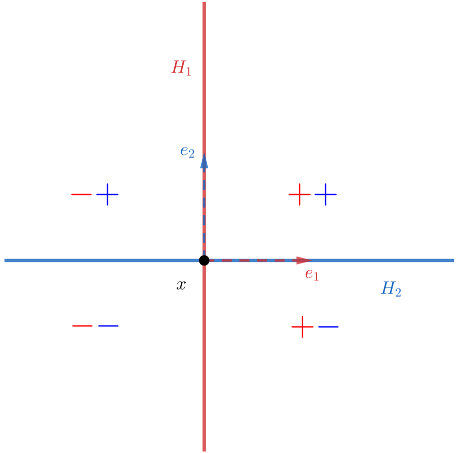

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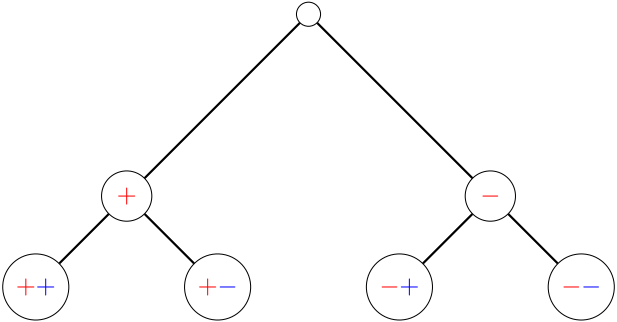

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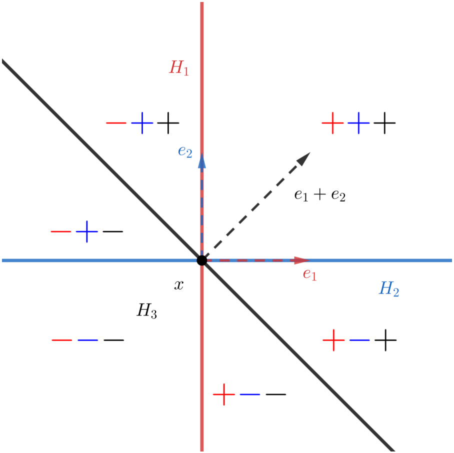

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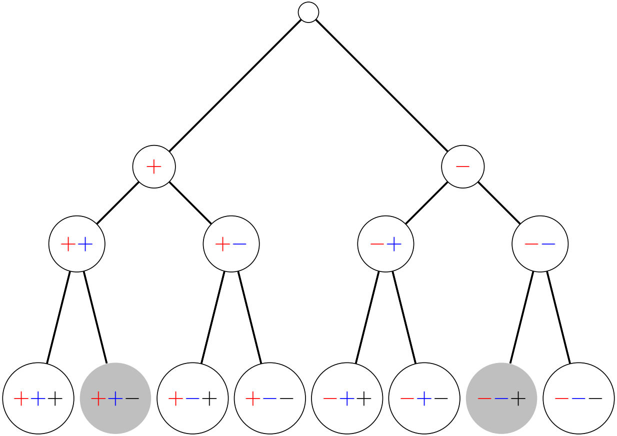
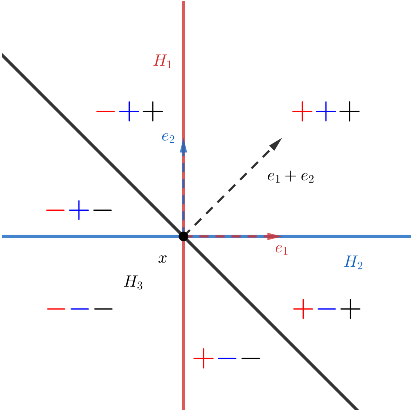


Plan

Infeasibility, matroids and circuits - 1



$++-$ (and $--+$) corresponds to an empty region: $+$ means right to H_1, $+$ over H_2, $-$ down left H_3: such a point does not exist.

Infeasibility, matroids and circuits - 2

With $p > 3$, $++-\bullet\ \bullet\ldots\bullet\ \bullet$ always infeasible.

Gordan's alternative

$M \in \mathbb{R}^{p\times n}$, exactly one is true:

$$\begin{cases} \exists\, d \in \mathbb{R}^n : Md > 0_{\mathbb{R}^p} \\ \exists\, \gamma \in \mathbb{R}^p_+ \backslash \{0\} : M^\top \gamma = 0 \end{cases} \quad (3)$$

$s \in \{\pm 1\}^p$ arbitrary:
$M = \mathrm{diag}(s)V^\top \to Md = (s_1 v_1^\top d; \ldots ; s_p v_p^\top d)$

"Feasibility of a system <u>or</u> element in the null space of the matrix"

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Infeasibility, matroids and circuits - 3

Instead: search for $\Gamma = \{\gamma\}$ and prune/stop the tree when an infeasibility is detected.
The γ's represent the "**circuits**" of the "**matroid**" defined by V

- the tree [from [RČ18]]
- slight improvements on the overall tree
- the dual algorithm detecting infeasibilities
- normal tree algorithm but with some infeasibility detection

Quite significant improvements brought by duality!

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Numerical results

	CPU times (in sec)						
	original	Improved tree-1		Improved tree-2		Fully dual tree	
Problem	code	Time	Ratio	Time	Ratio	Time	Ratio
rand-4-8-2	1.06	0.10	10.75	0.02	48.44	0.03	36.67
rand-7-9-4	1.13	0.45	2.51	0.29	3.95	0.02	68.67
rand-7-13-5	11.06	4.29	2.58	2.94	3.76	0.25	44.60
rand-8-15-7	64.79	29.53	2.19	27.59	2.35	4.54	14.29
rand-9-16-8	157.05	78.01	2.01	81.61	1.92	18.87	8.32
rand-10-17-9	352.42	196.09	1.80	213.48	1.65	70.19	5.02
srand-8-20-4	874.01	323.56	2.70	649.61	1.35	705.36	1.24
rc-2d-20-6	12.68	0.35	36.06	0.26	48.78	0.26	49.63
rc-2d-20-7	23.01	0.56	40.87	0.53	43.06	0.45	51.50
rc-perm-6	62.89	0.84	74.44	2.33	27.03	2.46	25.61
rc-perm-8	6589.31	85.70	76.89	1599.53	4.12	5290.13	1.25
rc-ratio-20-5-7	91.57	27.43	3.34	29.70	3.08	20.54	4.46
rc-ratio-20-5-9	88.24	25.21	3.50	27.54	3.20	17.75	4.97
rc-ratio-20-7-7	581.28	241.24	2.41	506.67	1.15	447.83	1.30
rc-ratio-20-7-9	460.64	162.95	2.83	315.67	1.46	234.72	1.96
Mean			16.60		13.90		30.31
Median			3.24		4.12		27.80

Conclusion

Duality and matroids brought considerable improvements

Future work

- adaptations to affine hyperplanes (for themselves)
- Julia code for this in development
- $\partial_B \min(\mathrm{nonlinear})$

Thank you for your attention! Any question?

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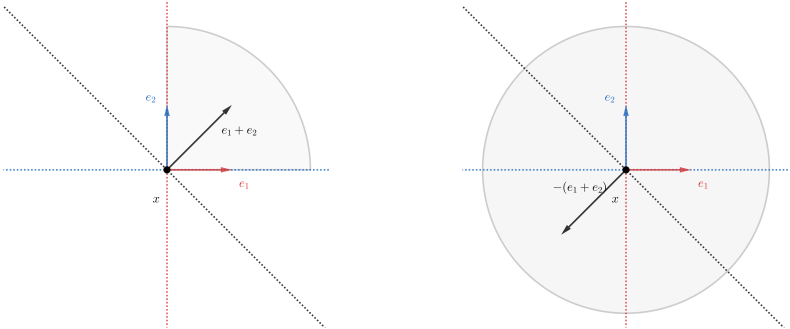
Various properties

-origin = center of symmetry ; $s_i v_i^\top d > 0 \Leftrightarrow (-s_i) v_i^\top (-d) > 0$



Feasible $\Leftrightarrow$ pointed cone   Infeasible $\Leftrightarrow$ non-pointed

-"connectedness" property (vertices = J's, edges = hyperplanes)

Method - adding vectors one at a time

With one more vector

- Given $(v_1, \ldots, v_{k-1})$; v_k ; $\mathcal{S}_{k-1} \subseteq \{\pm 1\}^{k-1}$

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Circuits of matroids

We look at subsets $I \subset [1:p]$, $\dim(\mathcal{N}(V_{:,I})) = \mathbf{1}$
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$$\dim(\mathcal{N}(V_{:,I})) = 1 \Rightarrow \mathcal{N}(V_{:,I}) = \mathrm{Vect}(\eta)$$
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$\mathcal{N}(V_{:,I})$ gives 'unsigned' η's which define the sign $s_J = 1$ because

if ≥ 2, smaller subsets are of $\dim(\mathcal{N}) = 1$

2^p LO feasibility $\leftrightarrow$ 2^p $\mathcal{N}$ searches; subsets of size $\leq 1 + \mathrm{rank}(V)$

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