A robust method for complementarity problems using a geometric viewpoint

Baptiste Plaquevent-Jourdain, with
Jean-Pierre Dussault, Université de Sherbrooke
Jean Charles Gilbert, INRIA Paris

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Outline

1. Setting
2. C-functions
   - Principle and examples
   - Beyond the local method
3. Newton-type methods and beyond
   - Newton-min
   - Polyhedral Newton-Min (PNM)
   - Least-squares over regularity
   - Complexity

Complementarity Problems (CPs)

General form: [FP03; AB08; CPS09]

Let $F, G : \mathbb{R}^n \mapsto \mathbb{R}^n$ be smooth,

$$\begin{aligned} \text{find } x \in \mathbb{R}^n \text{ s.t.} \quad & F(x) \geqslant 0,\ G(x) \geqslant 0,\ F(x)^\mathsf{T} G(x) = 0 \\ \iff \quad & 0 \leqslant F(x) \perp G(x) \geqslant 0. \end{aligned} \tag{1}$$

NCP(F)	$0 \leqslant x \perp F(x) \geqslant 0$
LCP(M, q)	$0 \leqslant x \perp (Mx + q) \geqslant 0$

At a solution x, $F_i(x) = 0$ or $G_i(x) = 0$ $\forall\ i \in [1 : n]$: 2^n choices; **combinatorial aspect**. LCPs are NP-hard [Chu89; Koj+91].

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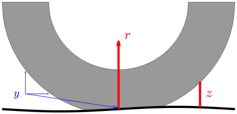
Relevance of CPs

Contact problems, threshold effects [HP90; FP97]



r: reaction, z: height

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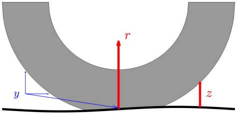

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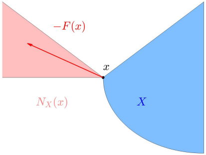
Constrained optimization:

$$\min \ f(x) \quad \text{s.t.} \quad g(x) \leqslant 0,$$

$$\text{KKT} \begin{cases} \nabla f(x) + \nabla g(x)\lambda = 0, \\ 0 \leqslant \lambda \perp (-g(x)) \geqslant 0. \end{cases}$$

VIs, generalized equations, MPECs MPCCs, [FP03; Rob80; HK09]...



Outline

Possible reformulation techniques

C-function:

$$\varphi : \mathbb{R}^2 \to \mathbb{R}, \ \text{s.t.} \quad \varphi(a, b) = 0 \quad \Longleftrightarrow \quad a \geqslant 0, b \geqslant 0, ab = 0$$

$$0 \leqslant F(x) \perp G(x) \geqslant 0 \ \Leftrightarrow \ H_\varphi(x) := (\varphi(F_i(x), G_i(x)))_{i \in [1:n]} = 0$$

$$\Leftrightarrow \ \min \theta := \frac{1}{2} ||H_\varphi(x)||^2$$

- $\varphi_{\mathrm{FB}}(a, b) := \sqrt{a^2 + b^2} - a - b$ [Fis92]
- $\varphi_{\min}(a, b) := \min(a, b)$, $H := H_{\min}$ [Pan90; Pan91; Qi93]
- [Fuk92; MS93; LT97; KYF97; Alc+20], surveys [FJ00; Gal12].

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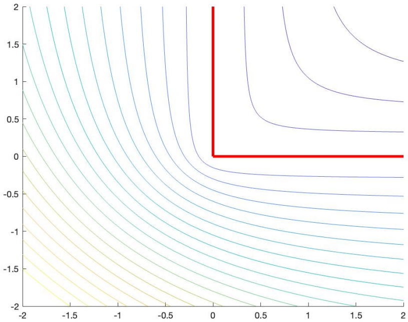
Illustration



Level sets of the C-function φ_{FB}, nondifferentiable only at the origin.
Level sets of the C-function $\varphi_{\min}$, nondifferentiable at the dashed line.
Red: 0-levelset, corresponding to solutions ($a \geqslant 0, b \geqslant 0, ab = 0$).

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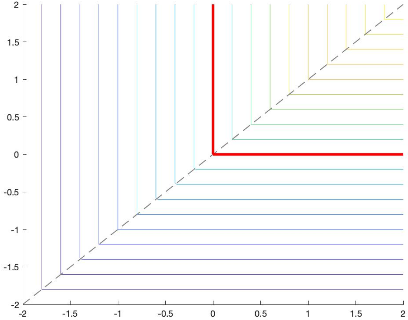

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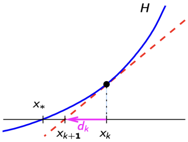

Smooth case

Iteration

$$x_{k+1} = x_k - H'(x_k)^{-1}H(x_k)$$
$$(= x_k + d_k)$$

x_0 near x^*, $H \in \mathcal{C}^{1,1}$,
$H'(x^*)$ nonsingular
$\Rightarrow$ convergence

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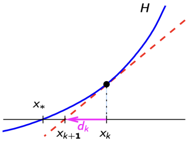

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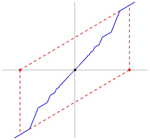

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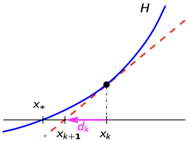

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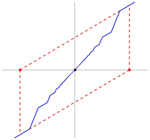

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Semismooth Newton iteration

$$x_{k+1} = x_k - J_k^{-1} H(x_k)$$
any $J_k \in \partial_{B|C} H(x_k)$

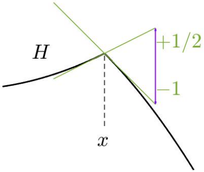
x_0 near x^*, H semismooth,
all $J \in \partial_{B|C} H(x^*)$ nonsingular
$\Rightarrow$ convergence

Generalizing derivatives

Two main differentials: [Cla90]

$$\partial_B H(x) := \{J \in \mathbb{R}^{n\times n} : \exists\, (x_k)_k \to x, x_k \in \overset{\text{domain}}{\mathcal{D}_H}, H'(x_k) \to J\},$$
$$\partial_C H(x) := \mathrm{conv}(\partial_B H(x)).$$

*B*ouligand, *C*larke (= generalizes the convex subdifferential)



1D example where $\partial_B H(x) = \{-1, +1/2\}$, $\partial_C H(x) = [-1, +1/2] \ni 0$!

Outline

Questions to clarify

1) x_0 near x^*? $\rightarrow$ $\min \theta := \frac{1}{2}||H_\varphi(x)||^2$

Case of FB (and many others)

$\theta_{\mathrm{FB}} := \frac{1}{2}||H_{\mathrm{FB}}||^2 = \frac{1}{2}\sum_{i=1}^{n} \varphi_{\mathrm{FB}}^2(\ldots_i)$ is smooth.

$-\nabla\theta_{\mathrm{FB}}$ is a descent direction, but slow: last resort [Mun+01].

2) Easy to get one J (suffices), but a good one?

Case of min

$\partial_B H(x)$ smaller, finite: less risk of singularity [IS14, p.151].

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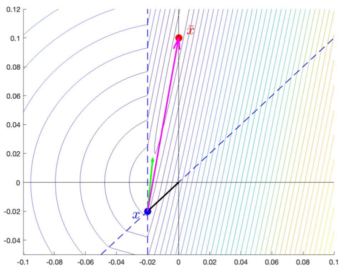
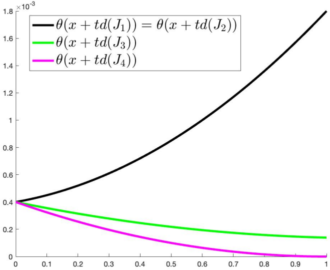
Globalization hindered by nonsmoothness

Example illustrated with the minimum C-function.



(a) Level sets of θ, nonsmooth at the dashed lines; the arrows are directions related to the J's.



(b) Depending on the J's, increase / decrease of θ (already observed in [Ben12]).

Trivia

Both have pros & cons: combine/adapt them
[DFK96; DFK00; PGP03; MS93; SQ99; CCK00; KK98; FP03].

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One interesting detour: [DGP25a]

for LCPs / affine F, G: $\partial_B H(x) \Leftrightarrow$ *hyperplane arrangement.*

Combinatorial geometry, since 19th century [Ste26; Rob87; Sch50].

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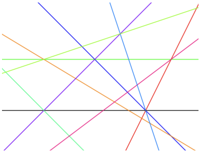

Identify the areas / intersections, #P-hard.

$\leftrightarrow$ matroids, zonotopes, posets, graphs...

Discussions in [DGP25b].

Outline

Recall of Newton-min (NM)

Problem and Newton-min algorithm ([Pan91; Qi93])

$$H(x) := \min(F(x), G(x)) = 0$$

- take a "good" x_0;
- $x_+ = x + d = x - J^{-1}H(x)$, $J \in \partial_B^{\times} H(x)$ (*) (or $\partial_C^{\times} H(x)$);
- requires all $J \in \partial_B^{\times} H(\bar{x})$ nonsingular.

$$\partial_B^{\times} H(x) := \partial_B H_1(x) \times \cdots \times \partial_B H_n(x)$$

$$(*) \quad d = -J^{-1}H(x) \iff H(x) + Jd = 0$$

Details of (*): index sets

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\begin{aligned}
&(1)\ F_i(x) < G_i(x), \quad H_i(x) = F_i(x), \quad \partial_B H_i(x) = \{F_i'(x)\},\\
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(1) & i \in \mathcal{F}(x) & J_{i,:} = F_i'(x) \\
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(3) & i \in \mathcal{E}(x) & J_{i,:} \in \{F_i'(x), G_i'(x)\}
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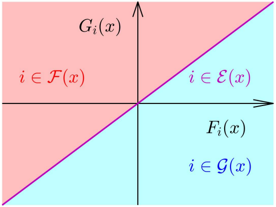

Representation of the index sets.

Newton-min (NM) system

For $i \in \mathcal{E}(x)$, linearize F_i or G_i?
$\mathcal{E}(x) = \mathcal{E}_{\mathcal{F}}(x) \cup \mathcal{E}_{\mathcal{G}}(x) \Leftrightarrow$ one J:

$$\begin{aligned} F_i(x) + F_i'(x)d &= 0 \quad i \in \mathcal{F}(x) \cup \mathcal{E}_{\mathcal{F}}(x) \\ G_i(x) + G_i'(x)d &= 0 \quad i \in \mathcal{G}(x) \cup \mathcal{E}_{\mathcal{G}}(x) \end{aligned}$$

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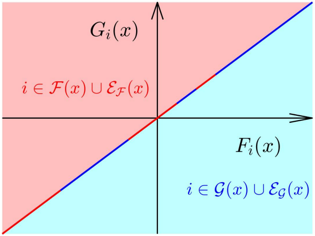


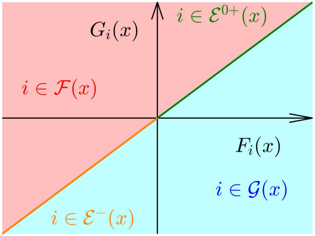
NM index sets.

In general, which partition?
$\rightarrow$ More elaborate system to decrease θ.

Outline

Starting point of PNM [DFG25]

$$\mathcal{E}^{0+}(x) := \{i : F_i(x) = G_i(x) \geqslant 0\}$$
$$\mathcal{E}^{-}(x) := \{i : F_i(x) = G_i(x) < 0\}$$



The “positive”/“negative” kinks.

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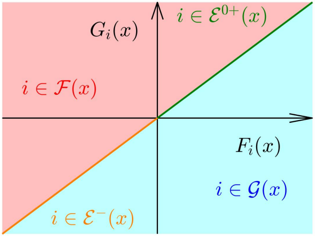

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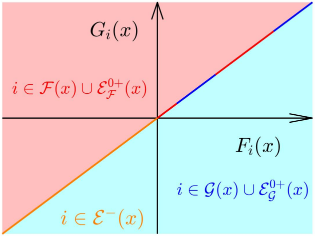

Type of PNM partitioning.

$\mathcal{E}^{0+}(x)$ violates complementarity, $\mathcal{E}^{-}(x)$ also violates $\geqslant 0$.
(See also [PG93; QS94; HPR92].)

The PNM method (main elements)

For $\mathcal{E}^{0+}_{\mathcal{F}}(x)$ and $\mathcal{E}^{0+}_{\mathcal{G}}(x)$, one equality, for $\mathcal{E}^{-}(x)$, **2** inequalities.

$$\text{polyhedron in } d : \begin{cases} F_i + F_i' d = 0 & i \in \mathcal{F}(x) \cup \mathcal{E}^{0+}_{\mathcal{F}}(x), \\ G_i + G_i' d = 0 & i \in \mathcal{G}(x) \cup \mathcal{E}^{0+}_{\mathcal{G}}(x), \\ F_i + F_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x), \\ G_i + G_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x). \end{cases}$$

such $d \Rightarrow \theta'(x; d) \leqslant -2\theta(x) < 0$, θ **decreases** along d.

PNM regularity condition

Around $\bar{x}$, $\forall\ \mathcal{E}^{0+}_{\mathcal{F}}(x) \cup \mathcal{E}^{0+}_{\mathcal{G}}(x)$ with x near $\bar{x}$, $\exists\ d(\mathcal{E}^{0+}_{\mathcal{F}}(x), \mathcal{E}^{0+}_{\mathcal{G}}(x))$.

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The PNM method (main elements)

For $\mathcal{E}^{0+}_{\mathcal{F}}(x)$ and $\mathcal{E}^{0+}_{\mathcal{G}}(x)$, one equality, for $\mathcal{E}^{-}(x)$, **2** inequalities.

$$\text{polyhedron in } d : \begin{cases} F_i + F_i' d = 0 & i \in \mathcal{F}(x) \cup \mathcal{E}^{0+}_{\mathcal{F}}(x), \\ G_i + G_i' d = 0 & i \in \mathcal{G}(x) \cup \mathcal{E}^{0+}_{\mathcal{G}}(x), \\ F_i + F_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x), \\ G_i + G_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x). \end{cases}$$

such $d \Rightarrow \theta'(x; d) \leqslant -2\theta(x) < 0$, θ **decreases** along d.

PNM regularity condition

Around $\bar{x}$, $\forall\ \mathcal{E}^{0+}_{\mathcal{F}}(x) \cup \mathcal{E}^{0+}_{\mathcal{G}}(x)$ with x near $\bar{x}$, $\exists\ d(\mathcal{E}^{0+}_{\mathcal{F}}(x), \mathcal{E}^{0+}_{\mathcal{G}}(x))$.

Outline

1. Setting
2. C-functions
   - Principle and examples
   - Beyond the local method
3. Newton-type methods and beyond
   - Newton-min
   - Polyhedral Newton-Min (PNM)
   - Least-squares over regularity
   - Complexity

What if the polyhedron is empty?

The problem is:

- nonsmooth,
- nonconvex,
- without (major) assumption.

Approach based on the structure of min.
Combinatorial nature, but no hypothesis on F and G.

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From the PNM system

Least-squares to find “a best possible d”, $\min ||(2)||^2/2$:

$$\begin{cases} F_i + F_i' d = 0 & i \in \mathcal{F}(x) \\ G_i + G_i' d = 0 & i \in \mathcal{G}(x) \\ F_i + F_i' d = 0 & i \in \mathcal{E}^{0+}_{\mathcal{F}}(x) & \gamma_i = 1, \overline{\gamma}_i = 0 \\ G_i + G_i' d = 0 & i \in \mathcal{E}^{0+}_{\mathcal{G}}(x) & \gamma_i = 0, \overline{\gamma}_i = 1 \\ F_i + F_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x) & \times \gamma_i \\ G_i + G_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x) & \times \overline{\gamma}_i \end{cases} \tag{2}$$

$\mathcal{E}^-(x)$ appears twice: **convex weights to balance**, $\gamma_- = (\gamma_i)_i \in [0,1]^{\mathcal{E}^-(x)}$ and $\overline{\gamma}_i := 1 - \gamma_i$.

Same for $\mathcal{E}^{0+}(x)$, $\gamma_+ = (\gamma_i)_i \in \{0,1\}^{\mathcal{E}^{0+}(x)}$, $\overline{\gamma}_i := 1 - \gamma_i$.
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Interest of the model and weights

$$\min_{d\in\mathbb{R}^n} \psi_{x,\gamma_+,\gamma_-}(d) := ||(2)||^2/2 \quad \text{with } \gamma_+, \gamma_- \tag{3}$$

- $\psi_{x,\gamma_+,\gamma_-}$ convex, smooth, piecewise quadratic model of θ at x
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Let γ_+, $\exists\ \gamma_-(\gamma_+)$ such that:

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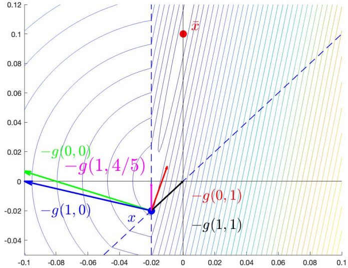
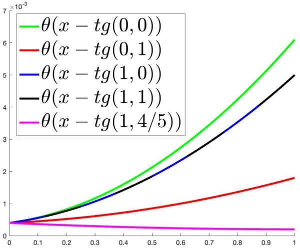
- if $g(\gamma_+,\gamma_-(\gamma_+)) = 0$, γ_+ irrelevant;
- wrong γ_- can make $\theta'(x;-g(\gamma_+,\gamma_-)) \geqslant 0$.

Illustration of the lemma





All partitions ($\{0,1\}^2$) increase θ. Correct weights yield descent.

Outline

Stationarity detection with the weights

We say x is strongly (Dini) stationary if $\forall\ d \in \mathbb{R}^n, \theta'(x; d) \geqslant 0$: 1^{st}-order order local optimality.

Stationarity with the γ's:

x strongly stationary $\Leftrightarrow \forall\ \gamma_+ \in [0,1]^{\mathcal{E}^{0+}(x)}, g(\gamma_+, \gamma_-(\gamma_+)) = 0\ (*)$

$\exists$ a descent direction $\Leftrightarrow$ **one good** γ_+ with $g(\gamma_+, \gamma_-(\gamma_+)) \neq 0$.

Complexity estimate via geometry

$M \in \mathbb{R}^{n\times q}$, $Z(M) := M[0,+1]^q = \{M\eta : 0_q \leqslant \eta \leqslant +1_q\} \subseteq \mathbb{R}^n$.
Centrally symmetric polytope, matrix $\times$ hypercube.
Combinatorics, control... [McM71; Zie07; Alt22; ST19]

$(*)$ is equivalent to verifying an inclusion between two zonotopes, which is co-NP-complete (in general) [KA21].

- γ_+ s.t. $g(\gamma_+, \gamma_-(\gamma_+)) \neq 0$, combinatorial in $|\mathcal{E}^{0+}(x)|$, $|\mathcal{E}^-(x)|$
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Into perspective

- PNM: assumption to make every γ_+ ✓.
- Weights: γ_+ bad if $g(\gamma_+, \gamma_-(\gamma_+)) = 0$.
- Good $\gamma_+ \Rightarrow g(\gamma_+, \gamma_-(\gamma_+)) \neq 0$: descent property.
- Optimal $\gamma_+ \Rightarrow g(\gamma_+, \gamma_-(\gamma_+)) \in \partial_C \theta(x) \setminus \{0\}$.
- If a method has: $g \to 0$, $x \to x^*$:
- by ∂_C, $0 \in \partial_C \theta(x^*)$ (weak stationarity).

Paradigm (see [BH20; Car+20; Car+21; JLZ22])

Knowing if x is strongly stationary is co-NP-complete,
obtaining one is much more difficult.

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Conclusion

Main take-aways

- H induces a geometric / combinatorial structure (zonotopes)
- explicitation of some results/properties,
- link between combinatorics and regularity assumptions.

Natural pursuits:

- numerical implementation;
- detailed convergence theorem;

Thank you for your attention! Any question?

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Smooth C-functions?

Yes! Mangasarian's framework [Man76]

Family of C-functions and easy condition to get smooth ones:

$$\begin{aligned} &\rho : \mathbb{R} \to \mathbb{R}, \quad \rho(0) = 0, \quad \rho \nearrow, \\ &\rho(|F_i(x) - G_i(x)|) - \rho(F_i(x)) - \rho(G_i(x)) = 0 \ \forall\ i \in [1:n]. \end{aligned} \tag{4}$$

But if $\bar{x}$ is not strictly complementary, then $\nabla H_\varphi(\bar{x})$ is singular: no fast local convergence [FP03, prop. 9.1.1, pp. 794-795].

Algorithm with nondegeneracy assumption in [Sub93].

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Summary of some basic methods

		$\partial_?$	reg(ularity) at $\bar{x}$	$\lvert\partial_?\rvert$	differentiable?
	NM	$\partial_B^\times H$	b-reg,	2^p J	piecewise
SNM	φ_{FB}	$\partial_C^{(\times)} H_{\mathrm{FB}}$	CD(FB)-reg,	© ("ball")	SC^1
		$\partial_B^{(\times)} H_{\mathrm{FB}}$	BD(FB)-reg,	© ("sphere")	
	min	$\partial_C^\times H$	CD(min)-reg,	© ("cube")	piecewise
		$\partial_B H$	smaller b-reg,	$\leqslant 2^p$ J	piecewise

Table: Properties of some nonsmooth methods. SNM: Semismooth Newton Method; NM: Newton-Min; © : continuum

$$(\partial_C^\times H(x) := \prod_{i=1}^{n} \partial_C H_i(x))$$

PNM computations

$$\theta'(x; d) = \sum_{i \in \mathcal{F}(x)} F_i(x) F_i'(x) d + \sum_{i \in \mathcal{G}(x)} G_i(x) G_i'(x) d$$
$$+ \sum_{i \in \mathcal{E}(x)} H_i(x) \min(F_i'(x) d, G_i'(x) d)$$

$$[d \text{ s.t.} \ldots] = -2\theta(x) \quad [\text{smooth}] \qquad [\downarrow \text{nonsmooth} \downarrow]$$

$$+ \sum_{i \in \mathcal{E}^{0+}(x)} \overbrace{H_i(x)}^{\geqslant 0} \overbrace{\min(F_i(x) + F_i'(x) d, G_i(x) + G_i'(x) d)}^{\text{one is } 0, \text{ so } \min \leqslant 0}$$

$$+ \sum_{i \in \mathcal{E}^{-}(x)} \overbrace{H_i(x)}^{<0} \overbrace{\min(F_i(x) + F_i'(x) d, G_i(x) + G_i'(x) d)}^{\text{both are } \geqslant 0, \text{ so } \min \geqslant 0}$$

$$= -2\theta(x) - \cdots - \cdots \leqslant -2\theta(x) \quad (< 0)$$

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Explicit computations

Let $\Gamma_+ = \mathrm{Diag}(\gamma_+), \Gamma_- = \mathrm{Diag}(\gamma_-)$

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$$\begin{aligned}
\theta'(x; -g(\gamma_+, \gamma_-)) = -||g(\gamma_+, \gamma_-)||^2 \\
+ H^{\mathsf{T}}_{\mathcal{E}^{0+}(x)}[\min(-F'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-), -G'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-)) \\
\underbrace{+\Gamma_+ F'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-) + \overline{\Gamma}_+ G'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-)]} \\
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$$\min(-a, -b) + \gamma a + \overline{\gamma} b = \begin{cases} \overline{\gamma}(b-a) \leqslant 0 & \text{if } a \geqslant b \\ \gamma(a-b) \leqslant 0 & \text{if } a \leqslant b \end{cases}.$$

Computation of the weights

Goal: get a nonzero g

$$\max_{\gamma_+ \in [0,1]^{\mathcal{E}^{0+}(x)}} \min_{\gamma_- \in [0,1]^{\mathcal{E}^{-}(x)}} ||g(\gamma_+, \gamma_-)||^2/2$$

where $g(\gamma_+, \gamma_-) = g_0 + M_+\gamma_+ + M_-\gamma_-$.

min: $\gamma_-(\gamma_+)$ to descend; max: distance (convex) on hypercube: $\rightsquigarrow \{0,1\}^{\mathcal{E}^{0+}(x)}$ (partitions); inclusion of polytopes: vertices suffice.

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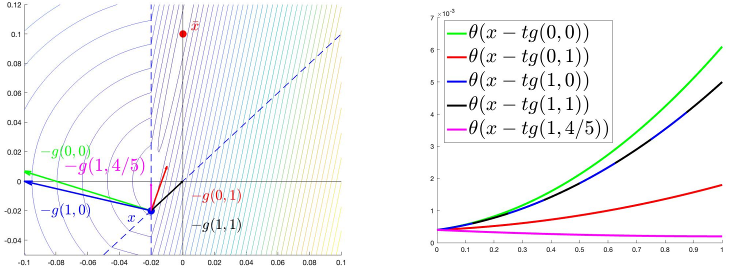
Example of weights computation – 1



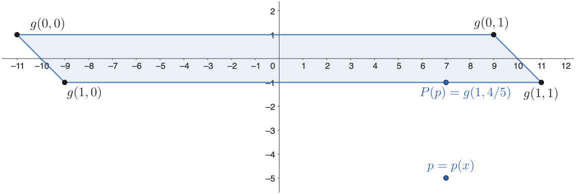
$$M = \begin{pmatrix} 1/2 & 1/2 \\ -5 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ -1/10 \end{pmatrix}, \quad x = \begin{pmatrix} -1/50 \\ -1/50 \end{pmatrix}$$

Here, $\mathcal{F}(x) = \varnothing = \mathcal{G}(x)$, $\mathcal{E}(x) = \{1, 2\}$. After computations. . .

Example of weights computation – 2

[up to factor $1/200$] The point to project is $(7;-5)(=g_1(x))$.

The zonotope to project on is $\begin{bmatrix} 1 & 10 \\ -1 & 0 \end{bmatrix} \times [-1,+1]^2$



The projection is $(7;-1) = 4/5 \times (11;-1) + 1/5 \times (-9;-1)$, and also $g(1,4/5) = 4/5g(1,1) + 1/5g(1,0)$

Some precisions for LCPs (1)

$$0 \leqslant F(x) = x \perp Mx + q = G(x) \geqslant 0$$
$$A = \mathcal{F}(x), \quad E = \mathcal{E}(x), \quad I = \mathcal{G}(x)$$

$$\mathcal{C}_{\text{ND}} \begin{cases} M_{I,I} \text{ nonsingular and} \\ M_{E,E} - M_{E,I} M_{I,I}^{-1} M_{I,E} \in \boldsymbol{ND} \end{cases}$$

Weighted framework for LCPs with $\mathcal{C}_{\text{ND}}$

The following properties are equivalent:

1) x is a solution,
2) for any (γ_+, γ_-), $g(\gamma_+, \gamma_-) = 0$,
3) for some $(\gamma_+, \gamma_-) \in \{0,1\}^{\mathcal{E}(x)}$, $g(\gamma_+, \gamma_-) = 0$.

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- For M a $\boldsymbol{P}$-matrix, $\mathcal{C}_{\mathrm{ND}}$ and $\mathcal{C}_{\mathrm{P}}$ ✓, solution ⇔ stationary,
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Algorithm properties

Sufficient decrease

1) $d_k(\lambda)/||d_k(\lambda)|| \underset{\lambda \to +\infty}{\to} -S_k^{-1} g_k/||S_k^{-1} g_k||$
2) for λ large enough, a descent formula holds

Convergence

Let (x_k, λ_k, S_k) be a sequence generated by algorithm **??**.

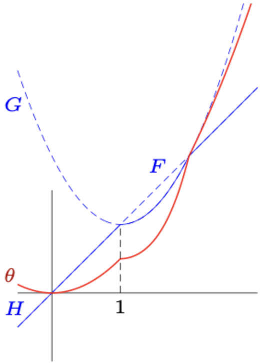
1) The sequence $(\theta(x_k))_k$ decreases thus converges.
2) If $(F'(x_k), G'(x_k), \lambda_k S_k)_{k \in \mathcal{K}}$ for a subsequence $\mathcal{K}$ is bounded, then $g_k \to 0$.

In particular, "good behavior" of algorithm is assumed: $\lambda_k \nrightarrow +\infty$.

"Concave kinks" – difficult (bad) limit points.

Consider a simple example with
$n = 1$, $F(x) = x$,
$G(x) = 1 + (x-1)^2$, $x_0 = 3/2$.

For $x \in (1, 2)$, $F(x) \neq G(x)$.



Counter-example

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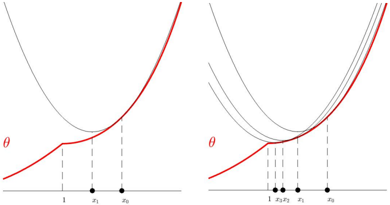
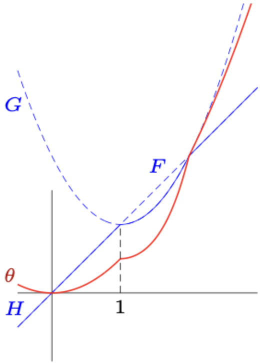


First iterates, convergence to $x = 1$. The black curves are the quadratic models ψ_x.



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