Primal and dual approaches for the enumeration of hyperplane arrangements

Baptiste Plaquevent-Jourdain, with
Jean-Pierre Dussault, Université de Sherbrooke
Jean Charles Gilbert, INRIA Paris

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Outline

Hyperplanes

Hyperplane H := affine (linear) subspace of dimension $n-1$ in $\mathbb{R}^n$.
For $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $H_{(v,t)} := \{x \in \mathbb{R}^n : v^\top x = \sum_{i=1}^n x_i v_i = t\}$.

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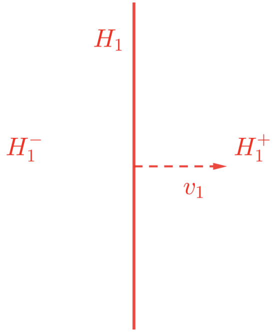
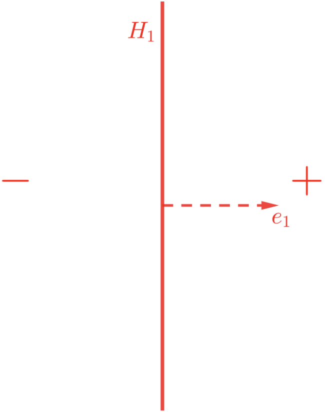


Several hyperplanes: geometric aspect



Example with a few hyperplanes and the *signs* of the halfspaces.

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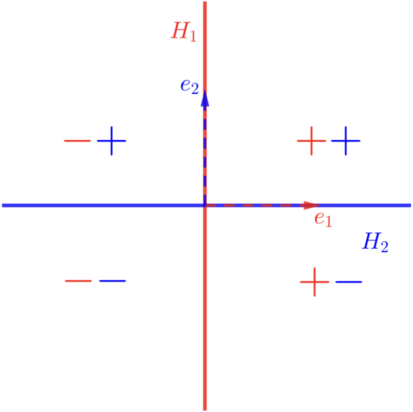

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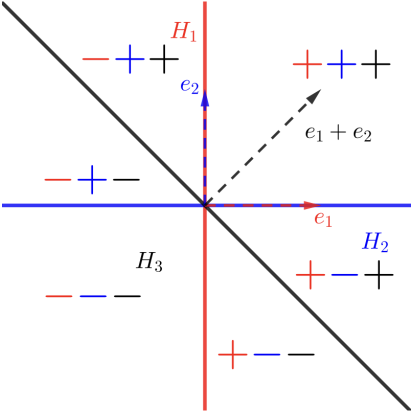
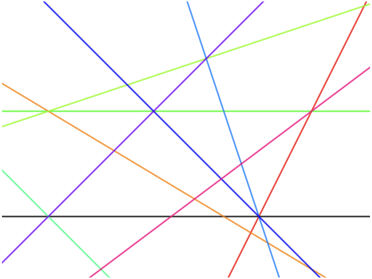


Example with a few hyperplanes and the *signs* of the halfspaces. The combinations of signs are **chambers** of the arrangement.

Less trivial example



More chaotic arrangement in dimension 2.

Already studied in the 19th century [Ste26; Rob87; Sch50].

Notation

Dimension $n \in \mathbb{N}^*$, $p \in \mathbb{N}^*$ hyperplanes, $v_i \in \mathbb{R}^n, \tau_i \in \mathbb{R}$ $1 \leqslant i \leqslant p$.

$$H_i := \{x \in \mathbb{R}^n : v_i^\mathsf{T} x = \tau_i\}, \quad V = [v_1 \ \ldots \ v_p], \quad \tau = [\tau_1; \ldots; \tau_p]$$

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Chambers: **subset** of the $\bigcap_{i=1}^{p} (H_i^+ \text{ or } H_i^-)$, the nonempty ones.

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Geometric to analytic: sign vectors

$$\text{find } \mathcal{S}(V, \tau) := \{s = (s_1, \ldots, s_p) \in \{\pm 1\}^p,$$
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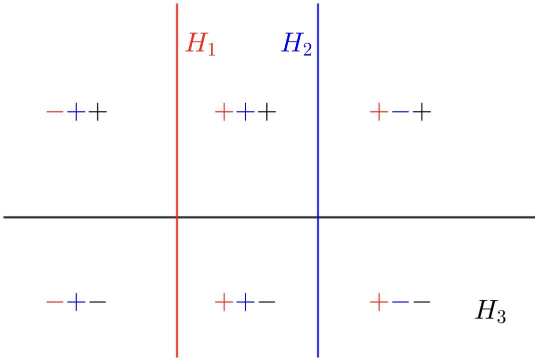
Subset of $\{\pm 1\}^p$; up to 2^p objects to identify.

Extension

The "whole arrangement" $\{\pm 1\} \to \{-1, 0, +1\}$: $\overline{\mathcal{S}}(V, \tau)$



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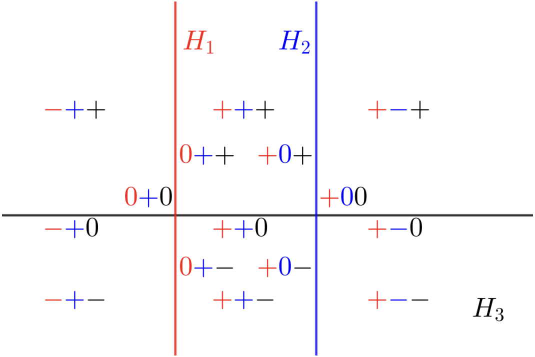

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Basic POVs

- geometric: hyperplane arrangements,
- algebraic: systems of affine inequalities.

- other algebra/geometry questions,
- nonsmooth analysis,
- convex analysis,
- computer science,
- some "applications".

Each viewpoint: new insights / tools / ...
Mostly $\tau = 0$: hyperplanes intersect in $0 \in \mathbb{R}^n$.

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Our initial motivation

Nonsmooth analysis/optimization

Not a single gradient (∇) but a set of *generalized gradients*.

- For a specific method in complementarity problems, the generalized gradient := the chambers of an arrangement.
- See [DGP25a], additional uses in [Pla25].

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Set of points $v_1, \ldots, v_p \subseteq \mathbb{R}^n$.
Up to 2^p ways to linearly separate them in two groups.

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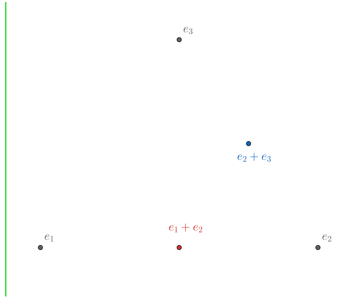

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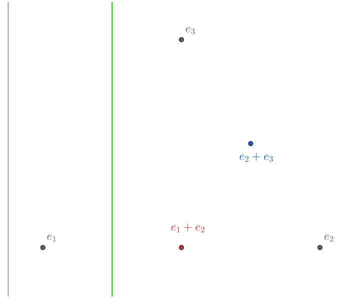

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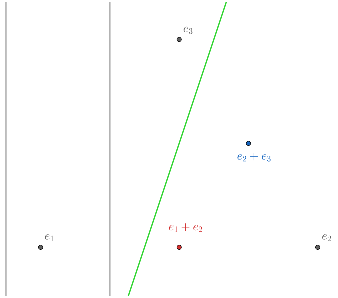

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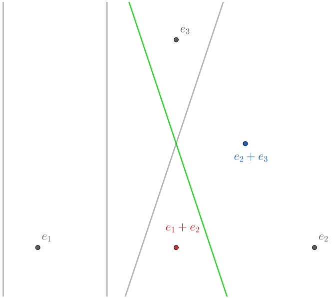

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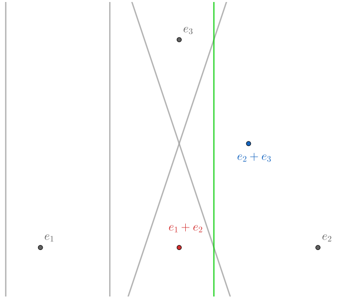

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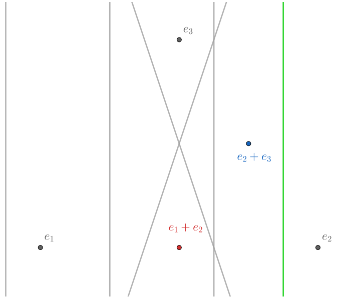

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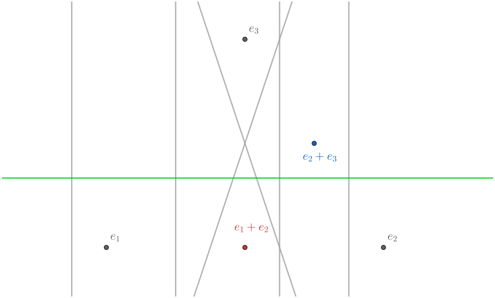

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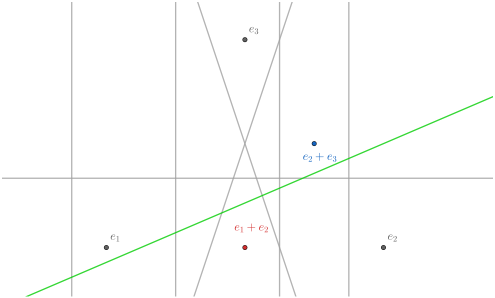

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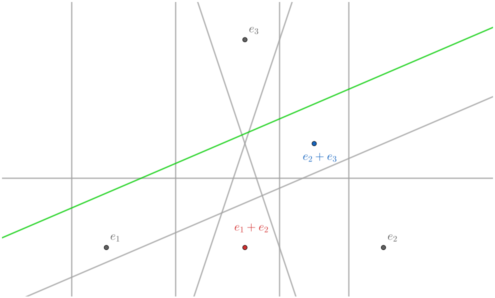

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Orientations of vectors forming cones (1) (see [DP22])

Let $v_1, \dots, v_p \subseteq \mathbb{R}^n$, $\mathrm{cone}\{v_1, \dots, v_p\} = \{\sum_{i=1}^{p} t_i v_i : t_i \geqslant 0\}$.

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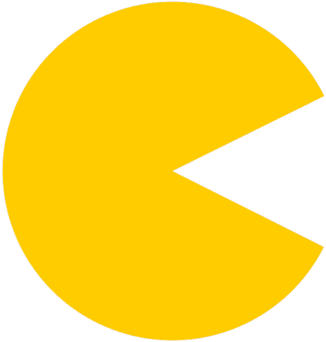

Pacman is **not pointed**: does not look like a cone. His mouth $\checkmark$.

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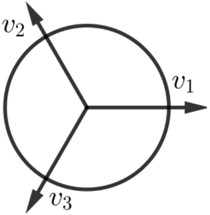

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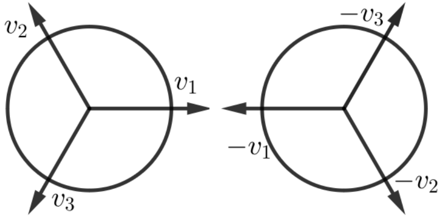
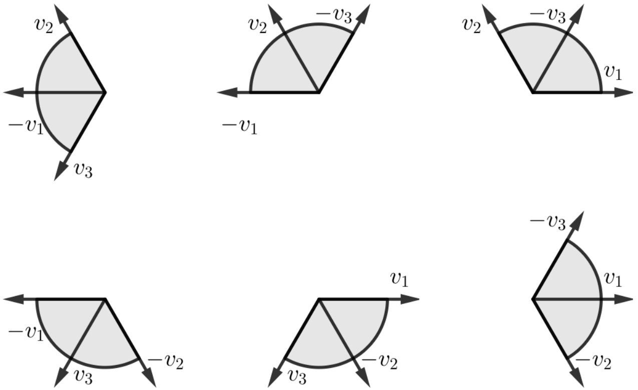


The cone of these vectors generate $\mathbb{R}^2$. The left cone $(+v_1, +v_2, +v_3)$ and the right cone $(-v_1, -v_2, -v_3)$ are not pointed.

Orientations of vectors forming cones (3)



Examples with pointed cones (swaps by opposing an extremal vector).
Here, $(+, +, +)$ and $(-, -, -)$ are incorrect, others are correct.

Orthants and null space

Orthant: the signs of $y \in \mathbb{R}^p$ remain constant.
Positive orthant $\mathbb{R}^p_{++} = \{y \in \mathbb{R}^p : y > 0\}$... 2^p orthants in total.

Duality

$$\mathcal{S}(V, 0) \iff \text{orthants of } \mathbb{R}^p \textbf{ not } \text{intersecting } \mathcal{N}(V).$$

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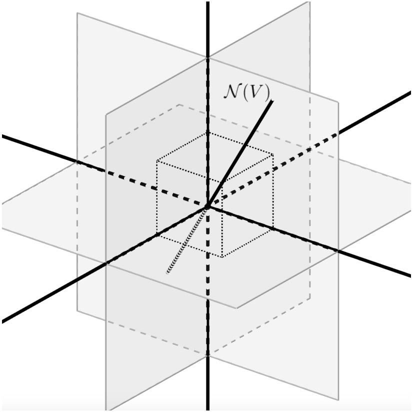
Orthants and null space, example



$\mathcal{N}(V)$ has nonempty intersection with orthants $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$, corresponding to infeasible $(+,+,+)$ and $(-,-,-)$.

Zonotopes

$V \in \mathbb{R}^{n\times p}$, $Z(V) := V[-1,+1]^p = \{V\eta : -1_p \leqslant \eta \leqslant 1_p\} \subseteq \mathbb{R}^n$.
Centrally symmetric polytope, [McM71; Zie07; Alt22; KA21; ST19]

Vertices: subset of the 2^p points $V\{-1,+1\}^p$: $Vs = \sum_{i=1}^p v_i s_i$, $s \in \{\pm 1\}^p$; some Vs are inside $Z(V)$: not vertices.

One main combinatorial properties of zonotopes

- $\mathcal{S}(V,0) \Leftrightarrow$ vertices of $Z(V)$;
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$\{\pm 1\}^p$: faces of dimension 0, $\{0,\pm 1\}^p$: faces of all dimensions.

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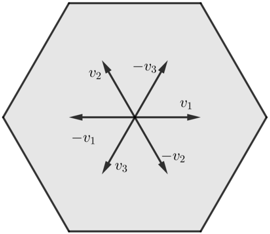

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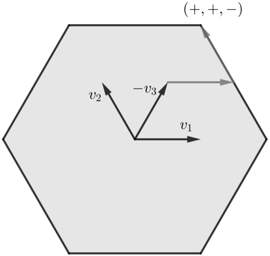

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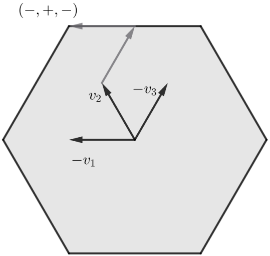

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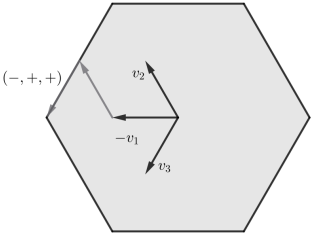

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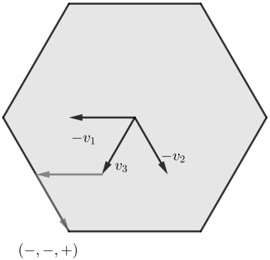

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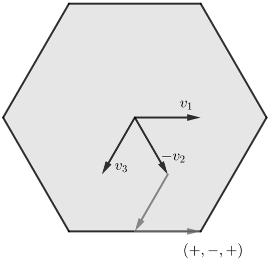

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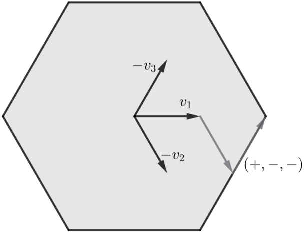

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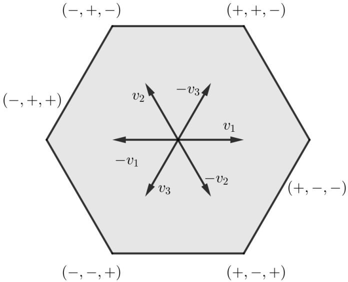

$(+,+,+)$ and $(-,-,-)$ do not correspond to vertices.

Arrangements and graphs [Sta07]

Graph G with vertices $= [1:n]$, p edges.
If $\{i,j\}$ is an edge, $H_{ij} := \{x : x_i - x_j = 0\}$ in the arrangement.

Relation with chambers

- An orientation of G is choosing $i \to j$ or $i \leftarrow j$ for each edge $\{i,j\}$: 2^p orientations.
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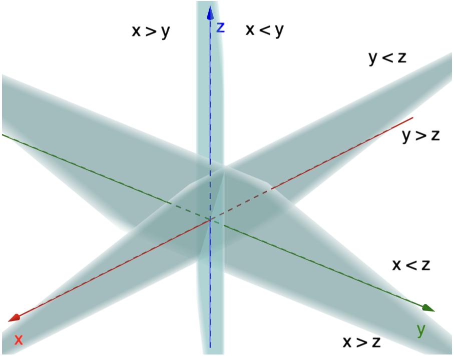

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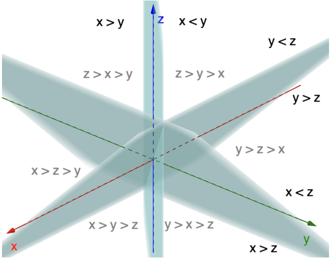

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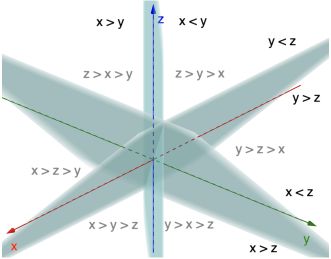

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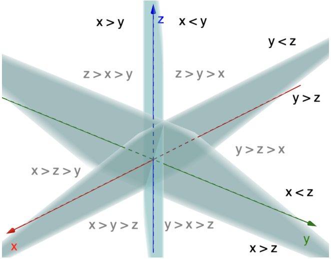

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Other combinatorial shenanigans [Sta07]

Very Important Property

The set of intersections of hyperplanes form a **poset**.

- structure “encoding” the combinatorics of the arrangement
- one level for $\mathbb{R}^n$, one level for planes, . . . , one level for points

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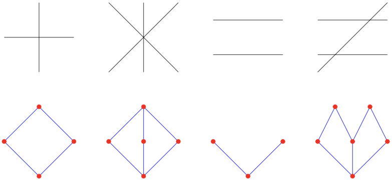

[Sta07, fig.2 p.8] Arrangements and corresponding posets. No signs $\pm$.

Robot path planning [Sle00]

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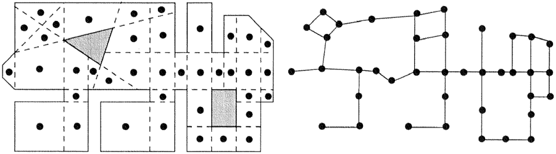

[Sle00, fig. 8.10, p.85] Chamber decomposition of a building.

One aspect of neural networks

Consider neurons with affine weights: $(v, t) \in \mathbb{R}^n \times \mathbb{R}$

$$\underbrace{x}_{\text{input}} \mapsto \underbrace{v^{\mathsf{T}} x - t}_{\text{action}} > 0?$$

Relation with arrangements

Each neuron creates a hyperplane in $\mathbb{R}^n$: layer = arrangement.

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Where some arrangements intervene

See [Win66; BEK23; Sta07; PS00; Ath96].

- specific families of arrangements (up to convention):
- combinatorics / geometry,
- algebraic statistics,
- quantum field theory,
- economics,
- psychometrics. . .

Applications with the whole arrangement $\{-1, 0, +1\}$ [EOS86].

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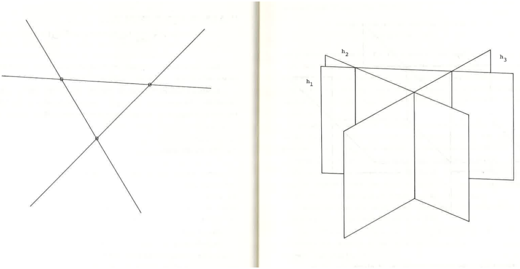

Projection for arrangements without full dimension [Zas75, fig. 2.1-2.2].

Formulas

$|\mathcal{S}(V,\tau)| \leqslant 2^p$; equality iff $p = \mathrm{rank}(V) = n$.

General upper bound ([Sch50], [Sta07])

$$|\mathcal{S}(V,\tau)| \leqslant \sum_{i=0}^{n} \binom{p}{i} \quad (\leqslant 2^p)$$

$=$ when in *general position*: $\simeq V$, τ random.

Formulas valid all the time (!): [Win66; Zas75]

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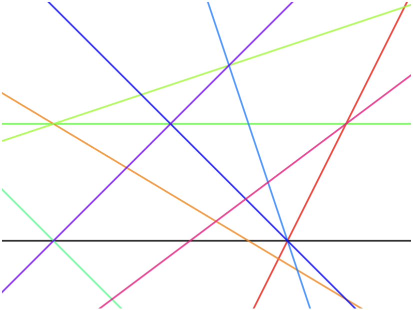

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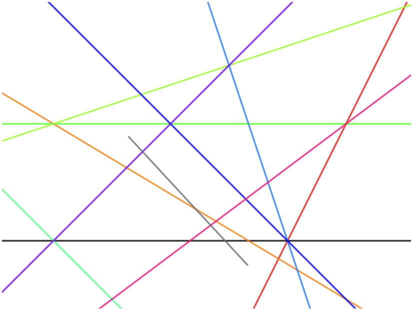

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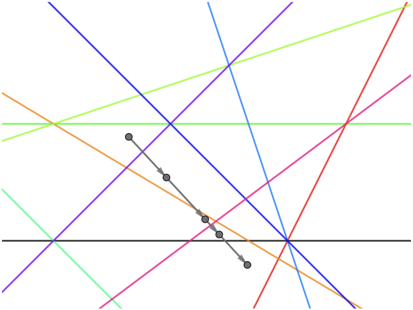
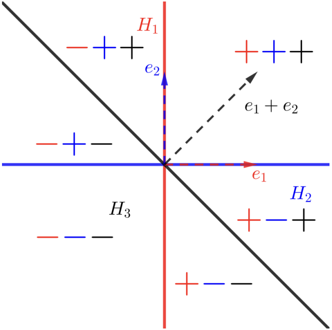


Paramount in some algorithms.
Transposable to vertices of zonotopes, cones. . .

Outline

Symmetric arrangements

$\mathcal{S}(V, 0)$ is symmetric, $0 \in \mathbb{R}^n$ center of symmetry.



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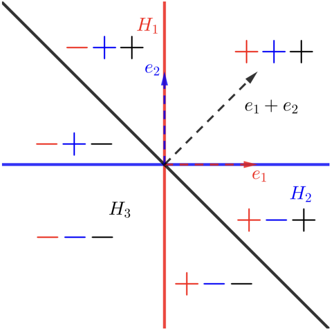

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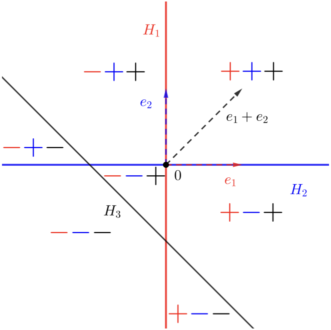

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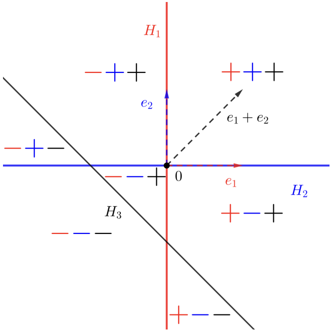

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Some software

- Sagemath (documentation) [Dev24]
- Macaulay2 (see for arrangements or matroids) [GS24]
- polymake [GJ00], see [KP20] for arrangements
- TOPCOM [Ram02; Ram23]
- for matroids: Oid, [KK05]
- see also OSCAR [Dec+24; OSC24] (used in [BEK23])

Warning

- Sometimes, theoretical algos (not always experimentations).
- Some may be lost to time (and/or not reimplemented?).

Two algorithms for the whole arrangement

Bieri-Nef [BN82], the first algorithm:

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Edelsbrunner-O'Rourke-Seidel [EOS86]

Asymptotic optimal complexity, *incremental* (H_1 then $H_2 \dots$). Involved algorithm: many definitions / subcases.

Back to the chambers: “simplex-type” algorithm

Chambers: connected graph but with **unknown nodes** and edges. Avis, Fukuda [AF92; AF96] (Sleumer [Sle98]) go through the graph *while* identifying the nodes := reverse search (RS).

Principle of the reverse search:

- start from an arbitrary chamber;
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RS is a general principle useful for other identification problems.

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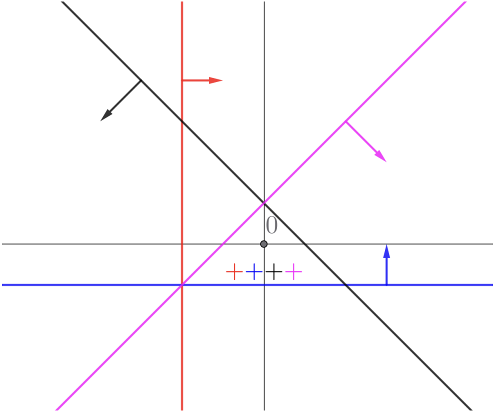

Grey arrows: order of the reverse search. Dotted grey arrows: pending neighbors; smaller ones: neighbors not visited due to the ordering rule(s).

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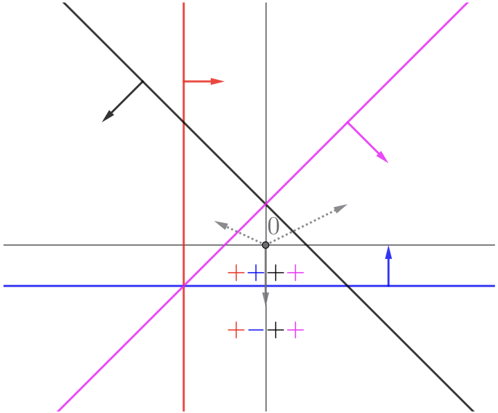

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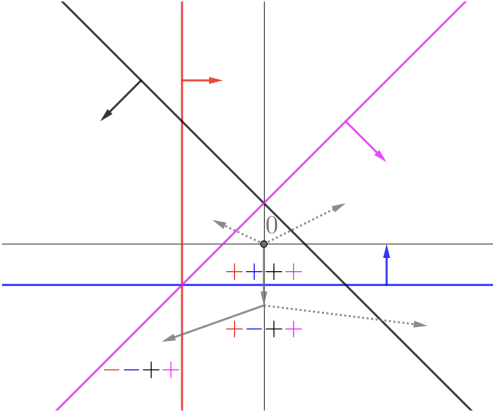

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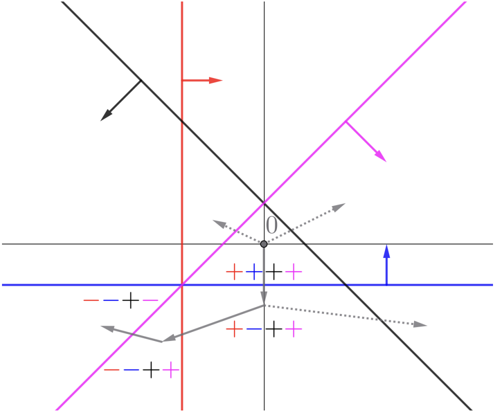

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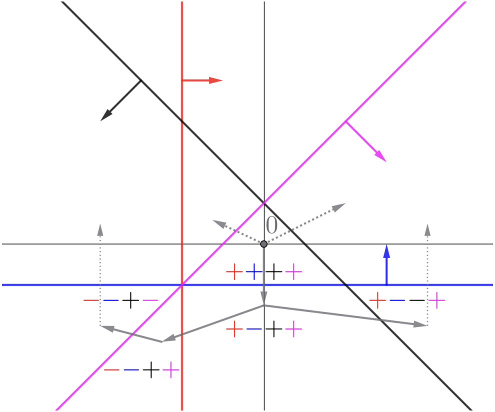

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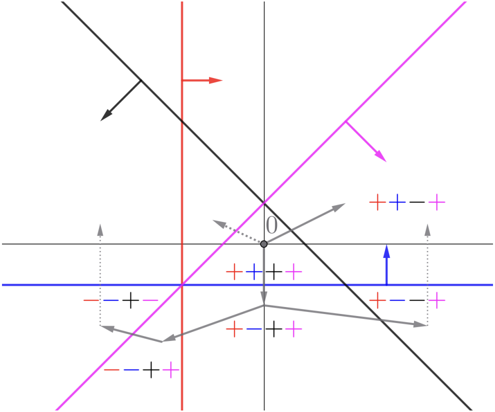

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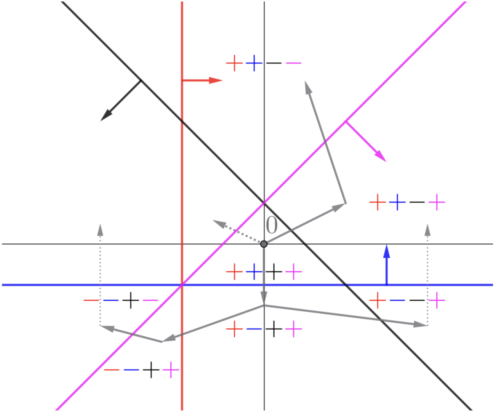

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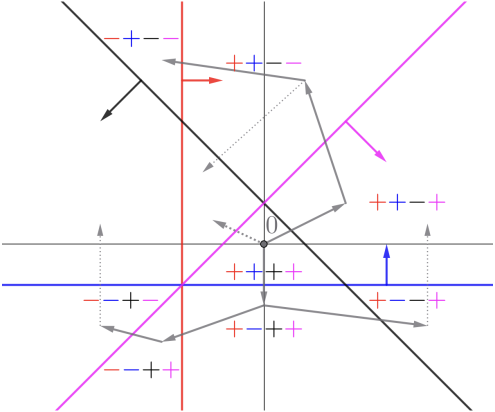

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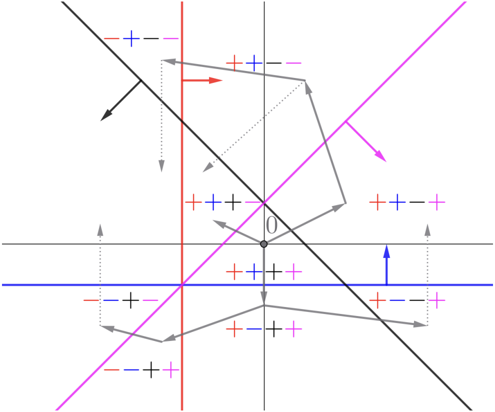

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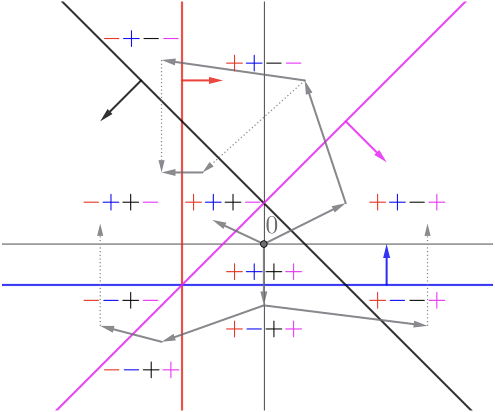

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For the zonotopes [DP22]

Recent method for the vertices of a zonotope.

Sort of revisited RS

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Rule to select only some potential neighbors.

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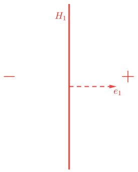
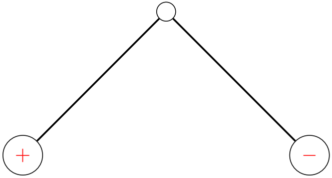
Rada-Černý algorithm: incremental tree





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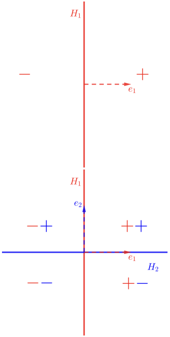

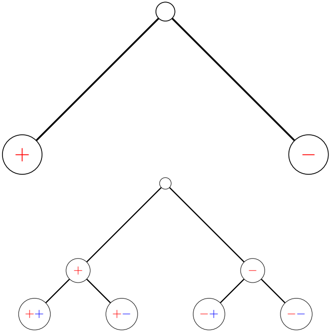

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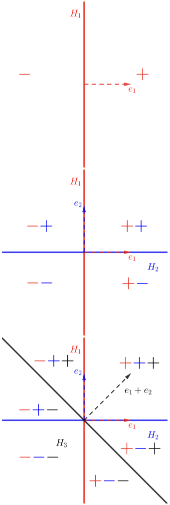

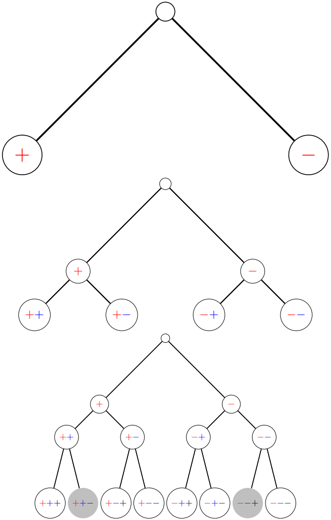

Main step at inner nodes

- node $s = (s_1, \ldots, s_k) \in \mathcal{S}(V_{:,[1:k]}, \tau_{[1:k]})$,
- with $x^s \in \mathbb{R}^n$: $s_i(v_i^\top x^s - \tau_i) > 0, \quad 1 \leqslant i \leqslant k$,
- $s_{k+1} := \text{sgn}(v_{k+1}^\top x^s - \tau_{k+1})$, $x^s \in H_{k+1}^{s_{k+1}}$: one descendant ✓
- for $(s, -s_{k+1})$: search for a solution x to

$$\exists\, x : \quad \begin{array}{r} s_i(v_i^\top x - \tau_i) > 0, \quad 1 \leqslant i \leqslant k \\ -s_{k+1}(v_{k+1}^\top x - \tau_{k+1}) > 0. \end{array} \qquad (1)$$

  Done by linear optimization.
- Better theoretically/practically than RS algorithms.

Main step at inner nodes

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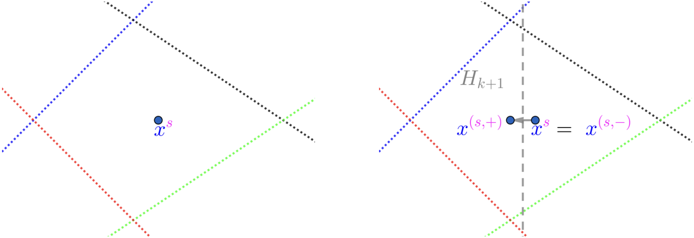
Closeness

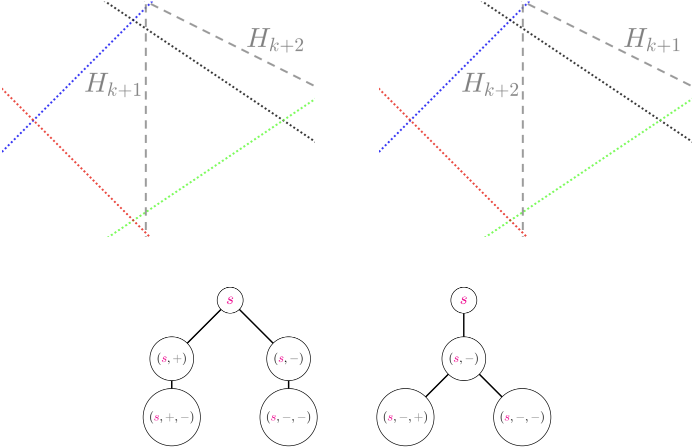


Left: level k. Right: shift of x^s when $x^s \underset{\sim}{\in} H_{k+1}$.

Details

For $s \in \{\pm 1\}^k$ with x^s, if $x^s \underset{\sim}{\in} H_{k+1} \Leftrightarrow v_{k+1}^\top x^s - \tau_{k+1} \simeq 0$, $(s, +1)$ and $(s, -1)$ in level $k+1$ without LOP.

Sequencing – which order to choose?



Changes inner levels – level p is always $\mathcal{S}(V,\tau)$.

A different approach

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Dual method: new viewpoint via convex duality

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- Which is *another* combinatorial problem.

That dual method is thus not that practical.
$\rightarrow$ primal-dual version, learns some infeasible combinations.
With everything, $\simeq$ 8 times faster [DGP25b].

Conclusion

Main take-aways

- relations/applications with many other topics
- various techniques can be employed for computations or to improve existing algorithms.

Thank you for your attention! Any question?

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General position expressions

All $\forall\ I \subseteq [1:p]$:

$$\begin{cases} \cap_{i\in I} H_i \neq \varnothing \text{ and } \dim(\cap_{i\in I} H_i) = n - |I| & \text{if } |I| \leqslant r \\ \cap_{i\in I} H_i = \varnothing & \text{if } |I| \geqslant r+1 \end{cases}$$

$$\begin{cases} \mathrm{rank}(V_{:,I}) = |I| & \text{if } |I| \leqslant r \\ \mathrm{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = r+1 & \text{if } |I| \geqslant r+1, \end{cases}$$

$$\begin{cases} \mathrm{rank}(V_{:,I}) = \min(|I|, r) \\ \mathrm{rank}([V;\tau^{\mathsf{T}}]_{:,I}) = \min(|I|, r+1). \end{cases}$$

Possible to slightly specify (simplify) when $\tau = 0$.

Affine $\leftrightarrow$ linear (1)

Main property (for instance [OT92])

Affine arrangements are “half” of linear arrangements.

Half of linear arrangement: half-space of *one* of the hyperplanes: $\mathbb{R}^n \to \{x \in \mathbb{R}^n : v_i^\top x > 0\}$.

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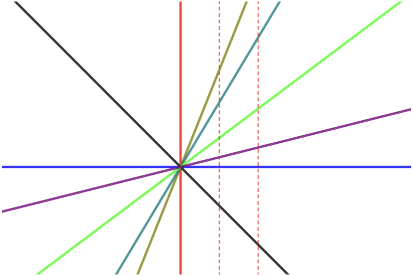

By homogeneity, translating H_i: dimension $n-1$, $p-1$ hyperplanes.

Affine ↔ linear (2)

One can to the converse to go from affine to linear by adding a dimension: $(V, \tau) \to (\mathcal{V}, 0)$

$$\mathcal{V} := \begin{bmatrix} V & 0 \\ \tau^\intercal & (\pm)1 \end{bmatrix}$$

$\mathcal{S}(V, \tau) :=$ affine$(n, p) \simeq$ linear$(n+1, p+1)$ (half of);
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Affine arrangements are slightly more general.

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The (other) augmented matrix

$\mathcal{V} = \begin{bmatrix} V & 0 \\ \tau^\top & -1 \end{bmatrix}$: to swap linear $\leftrightarrow$ affine, but useless "numerically".

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computing $\mathcal{S}(V,\tau)$ can be partially symmetrized (see later).

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Dual approach: avoid LOPs

- $s \in \{\pm 1\}^p$ is incompatible if $s \notin \mathcal{S}(V, \tau)$ ($s \in \mathcal{S}(V, \tau)^c$):

$$\nexists\, x \in \mathbb{R}^n : \qquad s \cdot (V^\mathsf{T} x - \tau) > 0,$$
$$\Leftrightarrow s \cdot V^\mathsf{T} x > s \cdot \tau.$$

- For $s \in \{\pm 1\}^p$ and $I \subseteq [1:p]$, s_I incompatible $\Rightarrow s$ is incompatible (more inequalities).
- For $s \in \{\pm 1\}^p$ incompatible, $\exists$ minimal incompatible s_I.
- With all incompatible s_I, no need for LO in the tree: check
- Find all the smallest I with s_I incompatible.

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Circuits and stem vectors – 1

A convex analysis tool: duality via Motzkin's alternative [Mot36]

$$\nexists\, x : Mx > m \quad \Longleftrightarrow \quad \exists\, \alpha \in \mathbb{R}^p_+ \setminus \{0\} : M^\mathsf{T}\alpha = 0, m^\mathsf{T}\alpha \geqslant 0.$$

$$s_I \text{ incompatible} \iff \nexists\, x \in \mathbb{R}^n : s_I \bullet V_{:,I}^\mathsf{T} x > s_I \bullet \tau_I$$

$$\iff \exists\, \alpha \in \mathbb{R}^I_+ \setminus \{0\} : V_{:,I}(\underbrace{s_I \bullet \alpha}_{=\eta \in \mathbb{R}^I}) = 0,\ \tau_I^\mathsf{T}(\underbrace{s_I \bullet \alpha}_{=\eta \in \mathbb{R}^I}) \geqslant 0.$$

The η is in $\mathcal{N}(V_{:,I}) \setminus \{0\}$, and oriented: $\tau_I^\mathsf{T}\eta \geqslant 0$ (otherwise: $-\eta$).

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The η is in $\mathcal{N}(V_{:,I}) \setminus \{0\}$, and oriented: $\tau_I^\mathsf{T}\eta \geqslant 0$ (otherwise: $-\eta$).

Circuits and stem vectors – 1

A convex analysis tool: duality via Motzkin's alternative [Mot36]

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Circuits and stem vectors – 2

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$$s \in \mathcal{S}(V,\tau)^c \quad \Longleftrightarrow \quad s_I \in \mathfrak{S}(V,\tau) \text{ for some } I \subseteq [1:p].$$

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Dual algorithm: tree with covering tests

- Compute $\mathfrak{S}(V,\tau)$ (via $\mathcal{C}(V)$).
- Test if $(s, +1)$ covers a stem vector.
- If yes, stop; if no, recursion on $(s, +1)$.
- Same for $(s, -1)$.

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Comparison

each inner node	Primal	Dual
verification concretely	1 LOP: low-dimension	1-2 covering test(s): array operations

Computing $\mathfrak{S}(V, \tau)$ is a combinatorial problem.
If $|\mathfrak{S}(V, \tau)|$ large, long computation and covering tests.

Intermediate: Primal-Dual, only *some* duality

- launch the primal tree;
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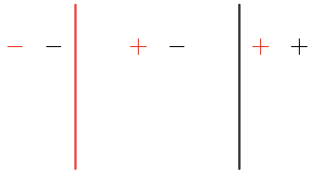
Illustration of duality

$M = s \cdot V^\mathsf{T}$, $m = s \cdot \tau$: $s \cdot (V^\mathsf{T} x - \tau) > 0 \Leftrightarrow s \cdot V^\mathsf{T} x > s \cdot \tau$



With $V = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\tau = [-1; 1]$, $\{x : x_1 = -1\}$ and $\{x : x_1 = +1\}$.
No $-+$ since (geometrically) $-$: left to the red hyperplane and $+$ right to the black hyperplane. Algebraically, $-$ means $x_1 < -1$ and $+$ $x_1 > 1$.

$$\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; (V \cdot [- \; +])\alpha = \begin{bmatrix} - & + \\ 0 & 0 \end{bmatrix} \alpha = 0, \; ([- \; +] \cdot \tau)\alpha = 2 \geqslant 0$$

About circuits/stem vectors

$$\mathcal{C}(V) := \{I \subseteq [1:p] : \mathrm{null}(V_{:,I}) = 1, \mathrm{null}(V_{:,I_0}) = 0 \ \forall\ I_0 \subsetneq I\}$$

No "good" algo (Rambau [Ram23]); adaptable for symmetries.
Upper bound $\binom{p}{r+1}$ [DSL06], $=$ under general position.
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For degenerate arrangements, short circuits so less susbets explored, but maybe lots of circuits (p large).
Ex: parallel hyperplanes – circuits of size 2 (so no larger subsets).

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Affine or linear?

coning/homogeneization/embedding/lifting/...

$$\mathcal{S}\left(\begin{bmatrix} V & 0 \\ \tau & -1 \end{bmatrix}, 0\right) = [\mathcal{S}(V,\tau) \times \{+1\}] \cup [-\mathcal{S}(V,\tau) \times \{-1\}],$$

i.e., “an affine arrangement in dimension n is the upper [or lower] half of a centered arrangement in dimension $n+1$”.

Natural way so swap between affine and linear arrangements
$\mathcal{S}(V,\tau) :=$ affine$(n, p) \simeq$ linear$(n+1, p+1)$ (half of);
$\mathcal{S}(V,0) :=$ linear$(n, p) \simeq$ affine$(n-1, p-1)$ (two opposite).

General improvement: “compaction”

Principle

- $\mathcal{S}(V, \tau)$ (and tree) asymmetric, we can “symmetrize”.
- For all variants (RČ, P, D, PD).

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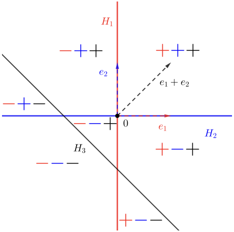
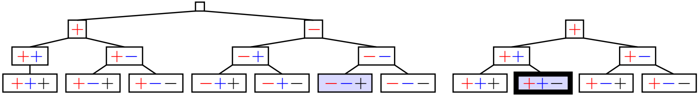


Asymmetric arrangement

$$\mathcal{S}(V, \tau) = \{(+++), (-++), (+-+), (--+), (---), (+--), (-+-)\}$$

except $(--+)$, rest symmetric

Compaction illustrated



Classic tree. Compact tree.

Blued nodes: asymmetric nodes, correction in the right tree. At the end, the other nodes are multipled by -1 to recover all nodes.

Details on compaction

$$\begin{cases} \mathcal{S}(V,0) & := \{s \in \{\pm 1\}^p : \exists\, x^s \in \mathbb{R}^n : s \bullet V^\mathsf{T} x^s > 0\} \\ \mathcal{S}(V,\tau) & := \{s \in \{\pm 1\}^p : \exists\, x^s \in \mathbb{R}^n : s \bullet (V^\mathsf{T} x^s - \tau) > 0\} \\ \mathcal{S}([V;\tau^\mathsf{T}],0) & := \{s \in \{\pm 1\}^p : \exists\, d^s \in \mathbb{R}^{n+1} : s \bullet [V^\mathsf{T}\ \tau] d^s > 0\} \end{cases}$$

$\mathcal{S}(V,\tau)$ has a *symmetric part* (not perfectly geometrically).

$\mathcal{S}(V,\tau)$ exactly between $\mathcal{S}(V,0)$ and $\mathcal{S}([V;\tau^\mathsf{T}],0)$ (symmetric).

Possible to quantify the difference in # of LOPs.

Compute less than $|\mathcal{S}(V,\tau)|$ chambers.

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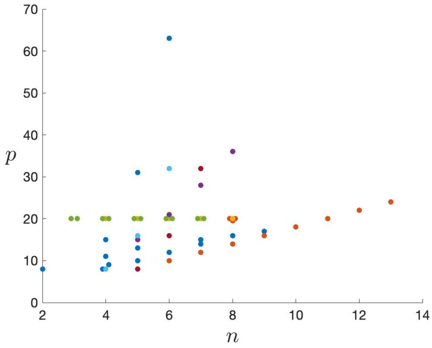
Algorithms and instances

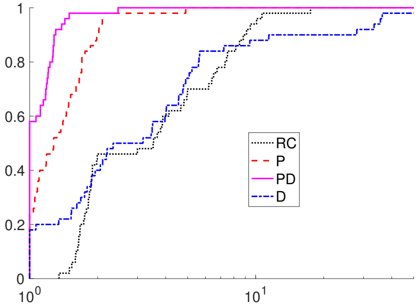
- Basic: [RČ18] – "RČ" (Rada Černý).
- With heuristics – "P" (Primal).
- Without LOPs, just stem vectors – "D" (Dual).
- LOPs and some stem vectors – "PD" (Primal-Dual).
- Relevance of compaction (/C).

Details on instances



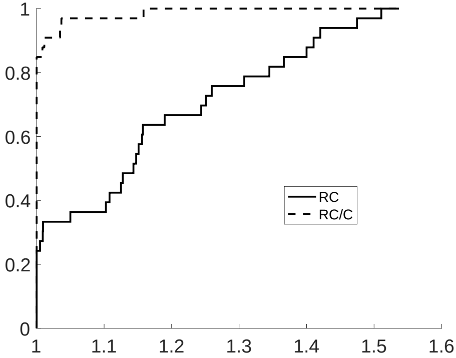
Pairs (n, p) for some linear and affine instances, grouped by colors. Instances up to 10^6 chambers/circuits (to run on a laptop). Example: $n = 7, p = 20$, up to 137980 chambers, 125970 stem vectors.

Comparison of the main variants



x-axis: relative efficiency (on time), y-axis: % of problems; above/left means being better. One has: primal-dual (PD) > primal (P) on some instances, both > Rada-Černý (RČ) and dual (D), which are quite close.

Variant vs compact variant



Compaction improves Rada-Černý RC, primal P and primal-dual PD (axis up to 2), but not really dual (D): less tests but more stems.

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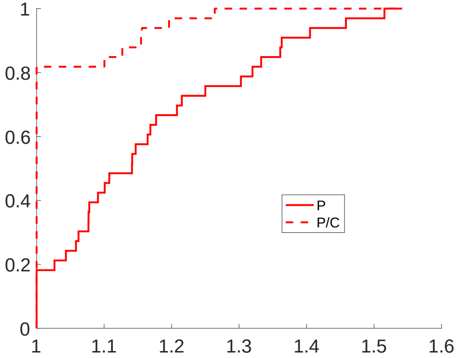

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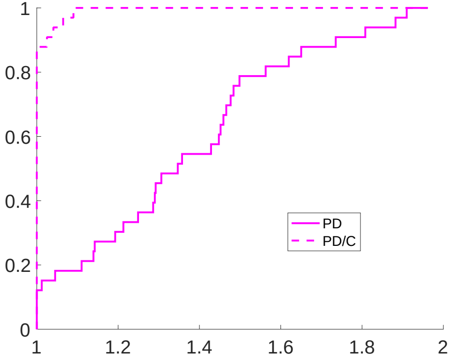

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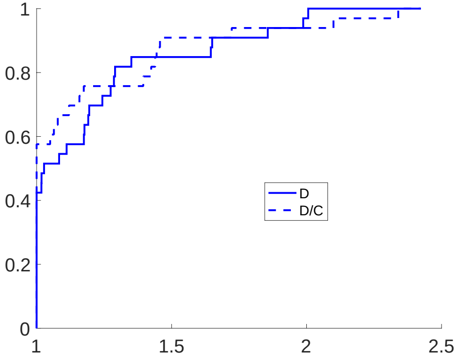

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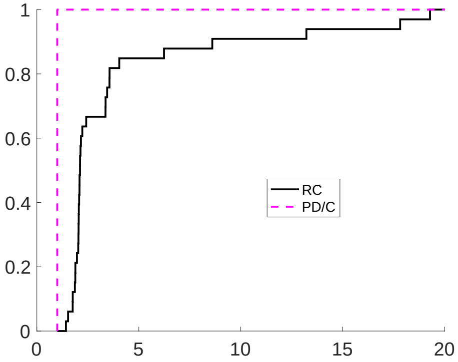
Initial vs best algorithm [not updated results]



Larger x-axis: average $\simeq 4$. Especially better on "structured" instances.

Possible code improvements: data structures, parallelism. . .

One last technique

Combinatorial symmetries

For instances where “all dimension are equivalent”, inspired from [BEK23] (just $|\mathcal{S}(V,\tau)|$) and [Ram23] ($\mathcal{C}(V)$ and other stuff).

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \tau = 0$$

Dimensions (rows) can be interchanged.

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Dimensions (rows) can be interchanged.
Idea: just consider a part of the tree (a part of the space), obtain the rest by combinatorial symmetry.

Such instances have interest for combinatoricians.

Illustration

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, p = 2^n - 1$$

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Same with 2 components $x_i < 0$, rest > 0, then 3 < 0...

The unifying method, Merino Mütze [MM24]?

$\{\pm 1\} \to \{0, 1\}$, *connected* vertices X of the hypercube.
A priori: the path may not be connected in $\mathbb{R}^n$;
To next chamber: binary variable, not LO

$$\min_{y,z} w^\top (y - x), \quad y_{P_0} = 0, \quad y_{P_1} = 1, \quad (2y - 1) \bullet (V^\top z - \tau) > 0?$$

For vertices of $P = \{z : Az \leqslant b\}$ **assumes it is a** $\mathrm{conv}(X)$ from A and b. (Not obvious according to Ziegler [Zie99]?)

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$$\min_{y,z} w^\top (y-x), \quad y_{P_0} = 0, \quad y_{P_1} = 1, \quad (2y-1) \bullet (V^\top z - \tau) > 0?$$

For vertices of $P = \{z : Az \leqslant b\}$ **assumes it is a** $\mathrm{conv}(X)$ from A and b. (Not obvious according to Ziegler [Zie99]?)

For circuits? $x(C)_i := \mathbb{1}(i \in C)$, $x(C) \in \{0,1\}^n$, $C(x) = \cup_{x_j=1}\{j\}$.
No "swaps" (flips) for circuits. The exchange axiom: 3 circuits...

$$\min_y w^\top (y-x), \quad y_{P_0} = 0, \quad y_{P_1} = 1, \quad \begin{cases} \mathrm{null}(V_{:,C(y)}) = 1, \\ \mathrm{null}(V_{:,C'}) = 0, C' \subsetneq C(y)? \end{cases}$$

Full arrangements: not only halfspaces

With $\{-1, 0, +1\}^p$, what changes? $2^p \to 3^p$,
known bounds (general position), RC algorithm with ternary tree.

Some things need to be adapted: especially compaction (relations).
Main issue: equalities ($s_i = 0$) are not maintained if $\tau \neq 0$.

For σ's, no changes? "chamber infeasible has no boundary":
so stem vectors $\sigma \in \{\pm 1\}^I$ mean every "$s^I \in [0, \sigma]$" infeasible too.

Algorithmically? Tree has 1 / 3 descendants (two ⇒ third).
Compute chambers, join neighbors for $n - 1$ subchambers, $n - 2$...
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About duality

- one stem vector ⇔ empty region $s \in \{-1, +1\}^J$ in the subarrangement with J;
- that empty region has no boundary (or empty boundary);
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Applications for the whole arrangement [EOS86]

theoretical / for complexity results, lots of “we present in dimension 2/3 and generalizations are clearly straightforward”

- “n-dimensional sorting”, “λ-matrices”
- K-Voronoi diagrams (points closest to a subsets of K points instead of just one)
- Testing degeneracy / general position (see later).
- Among p points, find the $n+1$ forming the smallest (in measure) simplex.

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Some properties

- up to 3^p sign vectors (objects) to identify,
- similar bounds in general position: for cells of dimension $k \in [0:n]$, $\binom{p}{n-k}\sum_{i=0}^{k}\binom{p-n+k}{i}$,
- formulas exists but more complicated,
- some symmetry properties hold,
- but not all: $\mathcal{S}_s(V,\tau) \neq \mathcal{S}(V,0)$: $\mathcal{S}(V,0)$ is centered so contains $(0,\ldots,0)$, which isn't $\mathcal{S}(V,\tau)$ unless centered,
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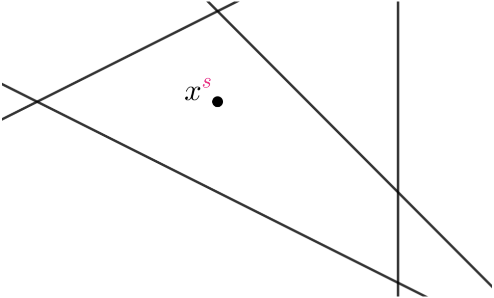

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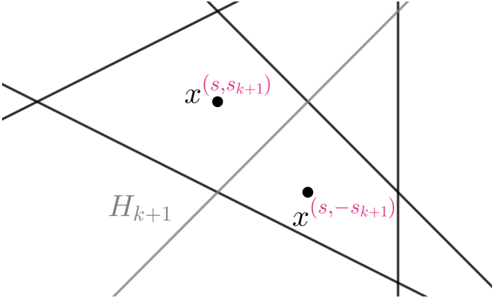

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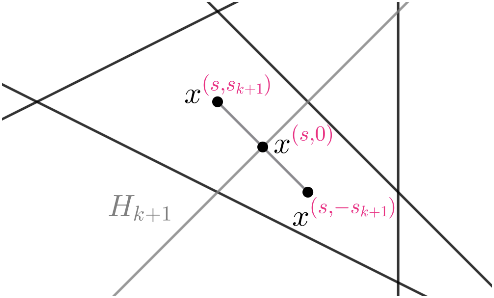

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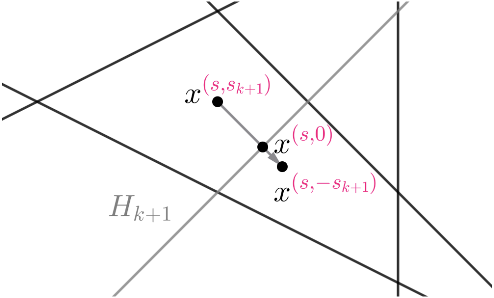

Variations of the algorithm

Idea 1: project the data in the subspaces (the $s_i = 0$)

Reduce the size of the LOPs, but chaining projections may be bad for precision / redundancy / ...

Idea 2: compute the "intersections"

Compute the nonempty $H_K := \cap_{k \in K} H_k$, project the hyperplanes H_i, $i \notin K$ in the subspace H_K then launch a smaller RČ in each.

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With the n-dimensional chambers, combine the neighbors to get the ones of dimension $n-1$, then $n-2$...

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