A robust method for complementarity problems using a geometric viewpoint

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Outline

Complementarity Problems (CPs)

General form: (Facchinei Pang [FP03], Acary Brogliato [AB08]. . .)

Let $F, G : \mathbb{R}^n \mapsto \mathbb{R}^n$ be smooth,

$$\begin{aligned} \text{find } x \in \mathbb{R}^n \text{ s.t.} \quad & F(x) \geqslant 0,\ G(x) \geqslant 0,\ F(x)^{\mathsf{T}} G(x) = 0 \\ \iff \quad & 0 \leqslant F(x) \perp G(x) \geqslant 0. \end{aligned} \tag{1}$$

- NCP(F): $0 \leqslant x \perp F(x) \geqslant 0$;
- LCP(M, q): (Cottle Pang Stone [CPS09])

$$0 \leqslant x \perp (Mx + q) \geqslant 0. \tag{2}$$

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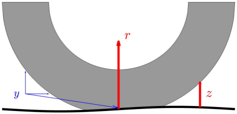
Relevance of CPs

Phenomena in competition, threshold effects [HP90; FP97]



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$$\forall \text{ point } y, \begin{cases} r(y) \geqslant 0, \\ z(y) \geqslant 0, \\ r(y)z(y) = 0. \end{cases}$$

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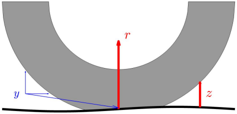

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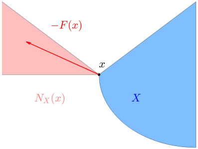
Constrained optimization:

$$\min \; f(x) \quad \text{s.t.} \quad g(x) \leqslant 0,$$

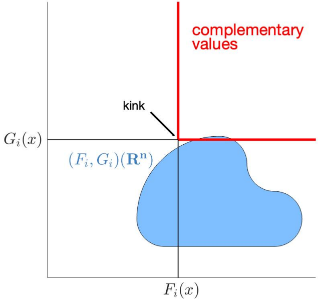
$$\text{KKT} \begin{cases} \nabla f(x) + \nabla g(x)\lambda = 0, \\ 0 \leqslant \lambda \perp (-g(x)) \geqslant 0. \end{cases}$$

Various other problems (Robinson [Rob80], [FP03]...).



An essential difficulty

At a solution x, $F_i(x) = 0$ or $G_i(x) = 0$ for all $i \in [1 : n]$:
2^n possibilities, **combinatorial aspect**.



NP-hard even for LCPs [Chu89], [Koj+91].

Some techniques

Pivoting [Lem65]: for LCPs, pivots between $x_i = 0$ or $(Mx + q)_i = 0$. But can visit all 2^n combinations [Mur78; Fat79].

Linearization [Jos79]: NCP $\rightarrow$ sequence of LCPs.

Interior points, smoothing [Koj+91; Had09; CNQ00; Vu+21]: $F_i(x) \geqslant 0$, $G_i(x) \geqslant 0$, $F_i(x)G_i(x) = 0 \rightarrow F_i(x)G_i(x) = \mu > 0$. Solution curve becomes smooth; different forms/approaches exist.

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Outline

Possible reformulation techniques

C-function:

$$\varphi : \mathbb{R}^2 \to \mathbb{R}, \text{ s.t. } \quad \varphi(a,b) = 0 \quad \begin{aligned} &\iff \quad a \geqslant 0, b \geqslant 0, ab = 0, \\ &\iff \quad 0 \leqslant a \perp b \geqslant 0. \end{aligned}$$

$$0 \leqslant F(x) \perp G(x) \geqslant 0 \;\Leftrightarrow\; H_\varphi(x) := (\varphi(F_i(x), G_i(x)))_{i\in[1:n]} = 0$$

- $\varphi_{\mathrm{FB}}(a,b) := \sqrt{a^2+b^2} - a - b$ ([Fis92])
- $\varphi_{\min}(a,b) := \min(a,b)$, $H := H_{\min}$ ([Pan90; Pan91; Qi93], see also [Kos76; Wie82])
- Framework of [LT97; KYF97], surveys [FJ00; Alc+20; Gal12].
- See also [Fuk92; MS93]

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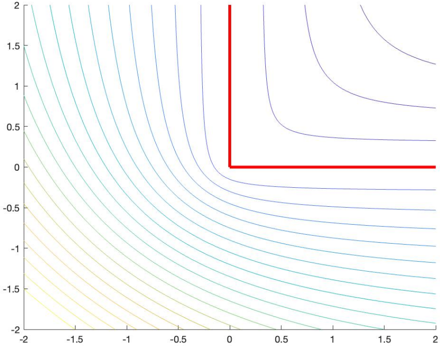
Illustration



Level sets of the C-function φ_{FB}, nondifferentiable only at the origin.
Level sets of the C-function $\varphi_{\min}$, nondifferentiable at the dashed line.

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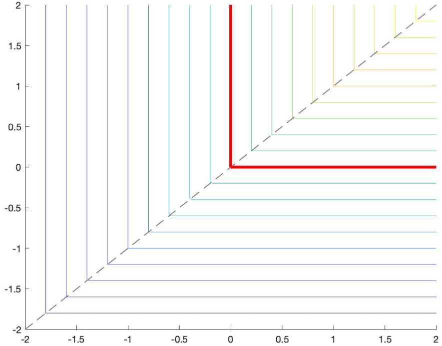

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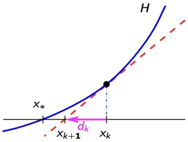
Systems of (nonsmooth) equations



Smooth case

Iteration

$$x_{k+1} = x_k - H'(x_k)^{-1} H(x_k)$$
$$(= x_k + d_k)$$

x_0 near x^*, $H \in \mathcal{C}^{1,1}$,
$H'(x^*)$ nonsingular
$\Rightarrow$ convergence

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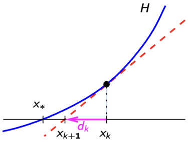

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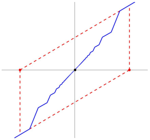

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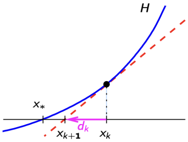

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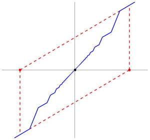

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Semismooth Newton iteration

$$x_{k+1} = x_k - J_k^{-1} H(x_k)$$
any $J_k \in \partial_{B|C} H(x_k)$

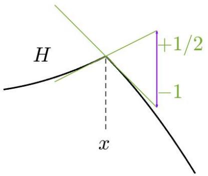
x_0 near x^*, H semismooth,
all $J \in \partial_{B|C} H(x^*)$ nonsingular
$\Rightarrow$ convergence

Generalizing derivatives

Two main differentials: [Cla90] (Clarke)

$$\partial_B H(x) := \{J \in \mathbb{R}^{n\times n} : \exists\ (x_k)_k \to x, x_k \in \overset{\text{domain}}{\mathcal{D}_H}, H'(x_k) \to J\},$$
$$\partial_C H(x) := \mathrm{conv}(\partial_B H(x)).$$

Bouligand, Clarke (= generalizes the convex subdifferential)



1D example where $\partial_B H(x) = \{-1, +1/2\}$, $\partial_C H(x) = [-1, +1/2] \ni 0$!

Smooth C-functions?

Yes! Mangasarian's framework [Man76]

Family of C-functions and easy condition to get smooth ones:

$$\begin{aligned} &\rho : \mathbb{R} \to \mathbb{R}, \quad \rho(0) = 0, \quad \rho \nearrow, \\ &\rho(|F_i(x) - G_i(x)|) - \rho(F_i(x)) - \rho(G_i(x)) = 0 \ \forall\ i \in [1:n]. \end{aligned} \tag{3}$$

But if x^* is not strictly complementary, then $\nabla H_\varphi(x^*)$ is singular: no fast local convergence [FP03, prop. 9.1.1, pp. 794-795].

Algorithm with nondegeneracy assumption in [Sub93].

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Globalization

How to get x_0 near x^*? $H(x) = 0 \to \min \theta := \frac{1}{2}||H(x)||^2$

Case of FB (and many others)

$\theta_{\mathrm{FB}} := \frac{1}{2}||H_{\mathrm{FB}}||^2 = \frac{1}{2}\sum_{i=1}^{n} \varphi_{\mathrm{FB}}^2(\ldots_i)$ is smooth.

Concretely, one can use $-\nabla\theta_{\mathrm{FB}}$ as a descent direction...
But $-\nabla$ is limited: often, a last resort (admittedly practical).

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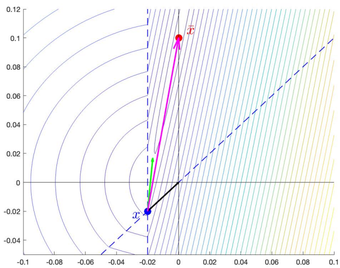
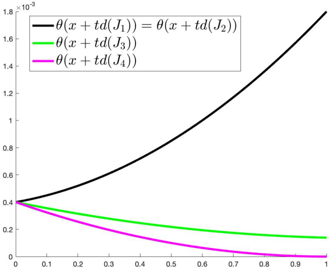
Globalization: example hindered by nonsmoothness



(a) Level sets of θ, nonsmooth at the dashed lines; the arrows are directions related to the J's.



(b) Depending on the J's, increase/decrease of θ (see also [Ben12]).

(Similar difficulty in MPECs, for instance [HK09, rem. 5.4 p. 888].)

FB vs minimum

- x_0 / globalization, FB because φ_{FB}^2 is smooth [Mun+01].
- J / ∂_B, minimum better: $\partial_B H(x)$ smaller [IS14, p.151] + finite termination if F and G affine.

- Numerically, no clear winner [DFK96; DFK00; PGP03];
- Combinations between both [KK98];
- adjust φ_{FB} [MS93; SQ99; CCK00] [FP03, fig. 9.1 p. 797]

How to choose J? Differentials are generally unknown.
Either just one J (suffices), or $\partial_B H(x) \subseteq$ [simple].

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One simple case

For LCP / linearizations of F, G

$$0 \leqslant Ax + a \perp Bx + b \geqslant 0 \;\Leftrightarrow\; H(x) = \min(Ax + a, Bx + b) = 0$$

$$\partial_B H(x) = \{J \in \mathbb{R}^{n \times n} : \exists\ (x_k)_k \to x, x_k \in \mathcal{D}_H, H'(x_k) \to J\}$$

- Related to hyperplane arrangements: [DGP25].
- Basic topic from combinatorial geometry, studied since early 19th century [Ste26; Rob87; Sch50].

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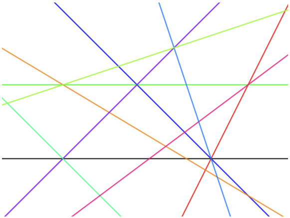
Quick comments



- $|\partial_B H(x)|$ is #P-hard; $\partial_B H(x)$ is at least as difficult.
- Related to: matroids, zonotopes, graphs, geometric / combinatorial notions posets... Families of arrangements...
- Heuristics and "primal-dual" technique (convex duality): state-of-the-art algorithm becomes $\simeq 8$ times faster.

Outline

Recall of Newton-min (NM)

Problem and Newton-min algorithm ([Pan91; Qi93])

$$H(x) := \min(F(x), G(x)) = 0$$

- take a "good" x_0;
- $x_+ = x + d = x - J^{-1}H(x)$, $J \in \partial_B^\times H(x)$ (or $\partial_C^\times H(x)$);
- requires all $J \in \partial_B^\times H(x^*)$ nonsingular.

$$\partial_B^\times H(x) := \partial_B H_1(x) \times \cdots \times \partial_B H_n(x)$$

$\partial_B H(x) \subseteq \partial_B^\times H(x) =$ simple; $\partial_{B(|C)} H$ not really necessary.

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Index sets

$$\begin{aligned}
&\text{if } F_i(x) < G_i(x), \quad H_i(x) = F_i(x), \quad \partial_B H_i(x) = \{F_i'(x)\},\\
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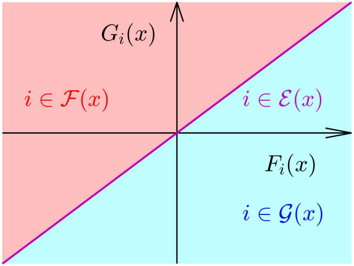


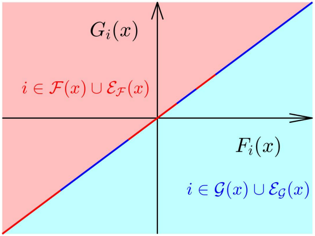
Representation of the index sets.

Newton-min system

NM splits $\mathcal{E}(x) = \mathcal{E}_{\mathcal{F}}(x) \cup \mathcal{E}_{\mathcal{G}}(x)$:

$$\begin{cases} (F(x) + F'(x)d)_{\mathcal{F}(x) \,\cup\, \mathcal{E}_{\mathcal{F}}(x)} & = 0, \\ (G(x) + G'(x)d)_{\mathcal{G}(x) \,\cup\, \mathcal{E}_{\mathcal{G}}(x)} & = 0, \end{cases}$$

simple, good local convergence.



NM index sets.

But which partition far from solutions?
$\to$ An augmented system ensuring descent (θ decreases).

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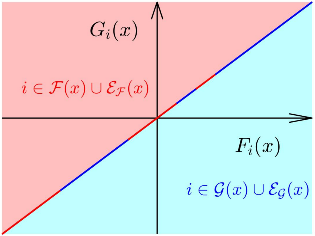

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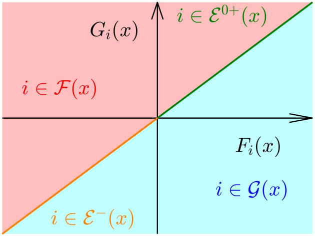
1. Introduction
2. C-functions
   - Principle and examples
   - Beyond the local method
3. Newton-type methods and beyond
   - Newton-min
   - Polyhedral Newton-Min (PNM)
   - Least-squares over regularity
   - Choice of the weights
   - Complexity and algorithm

Polyhedral Newton-min (PNM) [DFG25]

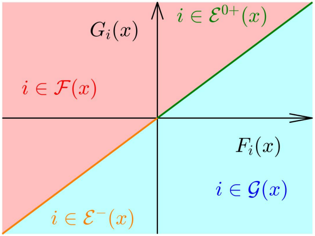
Let $\mathcal{E}^{0+}(x) := \{i : F_i = G_i \geqslant 0\}$ and $\mathcal{E}^{-}(x) := \{i : F_i = G_i < 0\}$,



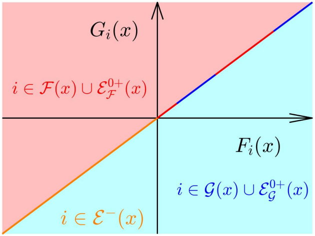
The "positive"/"negative" kinks.

Polyhedral Newton-min (PNM) [DFG25]

Let $\mathcal{E}^{0+}(x) := \{i : F_i = G_i \geqslant 0\}$ and $\mathcal{E}^{-}(x) := \{i : F_i = G_i < 0\}$, $\mathcal{E}^{0+}_{\mathcal{F}}(x) \cup \mathcal{E}^{0+}_{\mathcal{G}}(x)$ be a partition of $\mathcal{E}^{0+}(x)$:



The “positive”/“negative” kinks.



Type of PNM partitioning.

$\mathcal{E}^{0+}(x)$ violates complementarity, $\mathcal{E}^{-}(x)$ also violates $\geqslant 0$.
(See also [PG93; QS94; HPR92].)

Basic PNM method

For $\mathcal{E}_{\mathcal{F}}^{0+}(x)$ and $\mathcal{E}_{\mathcal{G}}^{0+}(x)$, one equality, for $\mathcal{E}^{-}(x)$, **2** inequalities.

$$\text{polyhedron in } d : \begin{cases} F_i + F_i' d = 0 & i \in \mathcal{F}(x) \cup \mathcal{E}_{\mathcal{F}}^{0+}(x), \\ G_i + G_i' d = 0 & i \in \mathcal{G}(x) \cup \mathcal{E}_{\mathcal{G}}^{0+}(x), \\ F_i + F_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x), \\ G_i + G_i' d \geqslant 0 & i \in \mathcal{E}^{-}(x). \end{cases}$$

1st-order info $\Rightarrow \theta'(x; d) \leqslant -2\theta(x) < 0$, θ **decreases** along d.

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PNM regularity

PNM regularity condition to have a d

At a solution x^*, for all partitions of $\mathcal{E}^{0+}(x)$ with x near x^*, a Mangasarian-Fromovitz condition holds at x^* with each partition.

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What if the polyhedron is empty?

The problem is:

- nonsmooth,
- nonconvex,
- without (major) assumption.

Method based on the structure of min.
Combinatorial nature, but no hypothesis on F and G.

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From the PNM system

Least-squares to find "a best possible d", $\min ||(4)||^2/2$:

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Indices of $\mathcal{E}^-(x)$ appear twice: **convex weights to balance**, $\gamma_- = (\gamma_i)_i \in [0,1]^{\mathcal{E}^-(x)}$ and $\overline{\gamma}_i := 1 - \gamma_i$.

Same for $\mathcal{E}^{0+}(x)$, $\gamma_+ = (\gamma_i)_i \in \{0,1\}^{\mathcal{E}^{0+}(x)}$, $\overline{\gamma}_i := 1 - \gamma_i$ $\{0,1\}$ for the partition, F_i or G_i.

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Brief comment on Levenberg-Marquardt (LM)

$$\min q(x) := \frac{1}{2}||r(x)||^2, \to \min_d \frac{1}{2}||r(x) + r'(x)d||^2 = q_x(d).$$

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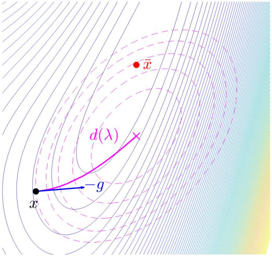

Smooth example. When $\lambda \searrow 0$, $d(\lambda) \to$ Newton direction, when $\lambda \to +\infty$: $d(\lambda)/||d(\lambda)|| \to -\nabla q(x)/||\nabla q(x)||$ the tangent of the curve.

Least-squares formulation

$q_x(d)/2 := ||(4)||^2/2$ with γ_+, γ_- + Levenberg-Marquardt

$$\min_{d \in \mathbb{R}^n} \psi_x(d) := q_x(d) + \frac{\lambda}{2} d^\mathsf{T} d, \quad \lambda \geqslant 0 \tag{5}$$

- ψ_x piecewise quadratic model of θ at x
- ψ_x **always has a minimizer** d even with empty polyhedron
- $g(\gamma_+, \gamma_-) := \nabla \psi_x(d = 0)$ has a descent property for θ?

$$g(\gamma_+, \gamma_-) = g_0(x) + M_+ \gamma_+ + M_- \gamma_-$$

Do γ_+, γ_- generate a good curve?

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Outline

Computations

Goal: decrease the nonsmooth merit function

Can we choose γ_+, γ_- to have $\theta'(x; -g(\gamma_+, \gamma_-)) \leqslant 0$?

$$\begin{aligned}\theta'(x; -g(\gamma_+, \gamma_-)) = \;& -\|g(\gamma_+, \gamma_-)\|^2 && \} \leqslant 0\\ & +\rho_{\mathcal{E}^{0+}(x)}(x, \gamma_+, g(\gamma_+, \gamma_-)) && \} \leqslant 0\\ & +\rho_{\mathcal{E}^{-}(x)}(x, \gamma_-, g(\gamma_+, \gamma_-)) && \} \geqslant 0\end{aligned}$$

$\rho_{\mathcal{E}^{0+}(x)}$, $\rho_{\mathcal{E}^{-}(x)}$ nonsmooth, quadratic in γ_+, γ_- respectively.

Choosing appropriate weights

Lemma:

Let γ_+, $\exists\ \gamma_-(\gamma_+)$ s.t. $\rho_{\mathcal{E}^-(x)}(x, \gamma_-(\gamma_+), g(\gamma_+, \gamma_-(\gamma_+))) = 0$:

$$\theta'(x; -g(\gamma_+, \gamma_-(\gamma_+))) = -\left\|g(\gamma_+, \gamma_-(\gamma_+))\right\|^2 + \rho_{\mathcal{E}^{0+}(x)}(x, \gamma_+, g(\gamma_+, \gamma_-(\gamma_+))) + 0 \leqslant 0.$$

- Possibly multiple $\gamma_-(\gamma_+)$, $\gamma_+ \mapsto g(\gamma_+, \gamma_-(\gamma_+))$ unique;
- if $g(\gamma_+, \gamma_-(\gamma_+)) = 0$, γ_+ irrelevant;
- **way to choose a good piece**,
- wrong γ_- can make $\theta'(x; -g(\gamma_+, \gamma_-)) \geqslant 0$.

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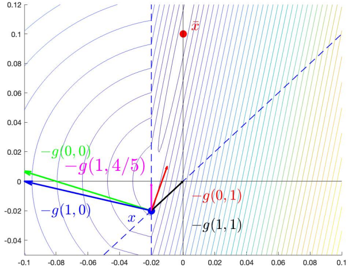
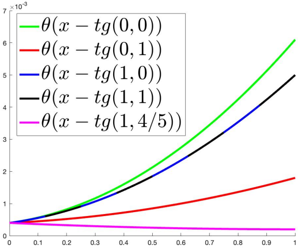
Illustration of the lemma





All partitions ($\{0,1\}^2$) increase θ. Correct weights yield descent.

Relation with PNM

PNM: assumption $\rightarrow$ every $\mathcal{E}_{\mathcal{F}}^{0+}(x) \cup \mathcal{E}_{\mathcal{G}}^{0+}(x)$ (γ_+) is convenient.
LM: one γ_- per γ_+ ($\mathcal{E}_{\mathcal{F}}^{0+}(x) \cup \mathcal{E}_{\mathcal{G}}^{0+}(x)$), bad if $g(\gamma_+, \gamma_-(\gamma_+)) = 0$.

Important link

For a given $\mathcal{E}_{\mathcal{F}}^{0+}(x) \cup \mathcal{E}_{\mathcal{G}}^{0+}(x)$, if there exists a d(PNM), there is NO γ_- s.t. $g = 0$ (every γ_- ✓).

Link based on convex duality [Mot36], weights "generalize" PNM.
Not an equivalence: possible to have no d(PNM) but weights work.

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- Robustification of the polyhedral system;
- "control" role for $\mathcal{E}^{0+}(x)$;
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Remaining questions:

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Stationarity detection with the weights

We say x is strongly (Dini) stationary if $\forall\ d \in \mathbb{R}^n, \theta'(x;d) \geqslant 0$: 1[st]-order order local optimality.

Stationarity with the γ's:

The following properties are equivalent:

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Complexity estimate via geometry

$M \in \mathbb{R}^{n\times q}$, $Z(M) := M[0,+1]^q = \{M\eta : 0_q \leqslant \eta \leqslant +1_q\} \subseteq \mathbb{R}^n$.
Centrally symmetric polytope, matrix $\times$ hypercube.
Combinatorics, control... [McM71; Zie07; Alt22; ST19]

2) is equivalent to verifying an inclusion between two zonotopes, which is co-NP-complete (in general) [KA21].

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An iteration of the algorithm

Algorithm Compute γ then usual LM

1: Obtain a suitable pair $\gamma_+, \gamma_-(\gamma_+)$ (or x^k stationary)
2: Get $d(\lambda^k) = \arg\min_d \psi_{x^k}(d)$ using $\lambda^k, \gamma_+, \gamma_-$
3: **while** $d(\lambda^k)$ not suitable **do**
4: increase λ^k and recompute $d(\lambda^k)$
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6: Update $x^{k+1} = x^k + d(\lambda^k)$ then λ^{k+1}

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- by ∂_C, if $x_k \to x^*$, $0 \in \partial_C \theta(x^*)$ (weak stationarity).

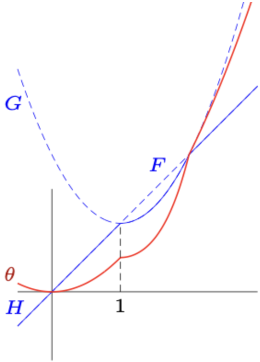
Paradigm (see [BH20; Car+20; Car+21; JLZ22])

Detecting a stationarity point is co-NP-complete,
obtaining one is much more difficult.

"Concave kinks" – difficult (bad) limit points.

Consider a simple example with $F(x) = x,\ G(x) = 1 + (x-1)^2$.

$x = 1$ is weakly stationary but not strongly (even less a solution)



Counter-example

Conclusion

Main take-aways

- H induces a geometric / combinatorial structure (zonotopes)
- explicitation of some results/properties.

Natural pursuits:

- numerical implementation;
- improved LM convergence theorem;
- $|F_i(x) - G_i(x)| < \tau$ instead of $F_i(x) = G_i(x)$.

Thank you for your attention! Any question?

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Summary of some basic methods

		$\partial_?$	reg(ularity) at x^*	$\|\partial_?\|$	differentiable?
	NM	$\partial_B^\times H$	b-reg,	2^p J	piecewise
SNM	φ_{FB}	$\partial_C^{(\times)} H_{\mathrm{FB}}$	CD(FB)-reg,	© (“ball”)	SC^1
		$\partial_B^{(\times)} H_{\mathrm{FB}}$	BD(FB)-reg,	© (“sphere”)	
	min	$\partial_C^\times H$	CD(min)-reg,	© (“cube”)	piecewise
		$\partial_B H$	smaller b-reg,	$\leqslant 2^p$ J	piecewise

Table: Properties of some nonsmooth methods. SNM: Semismooth Newton Method; NM: Newton-Min; © : continuum

$$(\partial_C^\times H(x) := \prod_{i=1}^{n} \partial_C H_i(x))$$

Computing the B-differential - multiple indices

$$H \text{ non} - \text{diff in } x_k \Leftrightarrow \exists\, i : (Ax_k + a)_i \overset{C1}{=} (Bx_k + b)_i, A_{i,:} \overset{C2}{\neq} B_{i,:}$$

$$I(x) := \{i \in [1:n] : A_{i,:}x + a_i = B_{i,:}x + b_i, A_{i,:} \neq B_{i,:}\}; |I(x)| = p$$

$$(Ax_k + a)_i \overset{C1}{=} (Bx_k + b)_i \overset{x_k = x + d_k}{\Leftrightarrow} (Ax + a)_i + A_{i,:}d_k = (Bx + b)_i + B_{i,:}d_k$$

$$\Leftrightarrow A_{i,:}d_k = B_{i,:}d_k \Leftrightarrow d_k \in v_i^{\perp}$$

$H_i := (B_{i,:} - A_{i,:})^{\perp} := v_i^{\perp}$, $v_i \neq 0$ ($C2$) ; for ∂_B, $\mathbb{R}^n \backslash (x + \cup_{i=1}^{p} H_i)$

$$\mathbb{R}^n = H_i^- \cup H_i \cup H_i^+, \quad \begin{cases} H_i^- = \{x \in \mathbb{R}^n : v_i^T x < 0\} \\ H_i^+ = \{x \in \mathbb{R}^n : v_i^T x > 0\} \end{cases}$$

$$H_i^- \Leftrightarrow B_{i,:}d - A_{i,:}d < 0 \Leftrightarrow \min(\dots) = (B\dots)_i \Leftrightarrow J_{i,:} = B_{i,:}$$
$$H_i^+ \Leftrightarrow B_{i,:}d - A_{i,:}d > 0 \Leftrightarrow \min(\dots) = (A\dots)_i \Leftrightarrow J_{i,:} = A_{i,:}$$

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$$\begin{aligned}(Ax_k + a)_i \overset{C1}{=} (Bx_k + b)_i \overset{x_k = x + d_k}{\Leftrightarrow} & (Ax + a)_i + A_{i,:}d_k \\ & = (Bx + b)_i + B_{i,:}d_k \\ & \Leftrightarrow\ A_{i,:}d_k = B_{i,:}d_k \Leftrightarrow d_k \in v_i^{\perp}\end{aligned}$$

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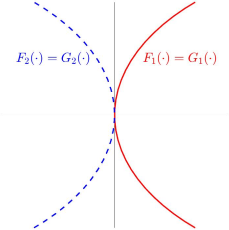
$$H_i^+ \Leftrightarrow B_{i,:}d - A_{i,:}d > 0 \Leftrightarrow \min(\dots) = (A\dots)_i \Leftrightarrow J_{i,:} = A_{i,:}$$

If functions are not affine

- Sets $\{y \text{ near } x : F_i(y) = G_i(y)\}$ are not hyperplanes.
- First order $F_i(y) = F_i(x) + F_i'(x)(y-x)$ (and G): $\rightarrow$ affine case with hyperplanes and yields a subset of the real $\partial_B H(x)$.
- Much harder due to “higher order”...



Here, linearizations yield the same hyperplane: two chambers, but there is a third from points between the two curves.

PNM computations

$$\theta'(x;d) = \sum_{i\in\mathcal{F}(x)} F_i(x)F_i'(x)d + \sum_{i\in\mathcal{G}(x)} G_i(x)G_i'(x)d$$

$$+ \sum_{i\in\mathcal{E}(x)} H_i(x)\min(F_i'(x)d, G_i'(x)d)$$

$$[d \text{ s.t.} \dots] = -2\theta(x) \quad [\text{smooth}] \qquad [\downarrow \text{nonsmooth} \downarrow]$$

$$+ \sum_{i\in\mathcal{E}^{0+}(x)} \overbrace{H_i(x)}^{\geqslant 0} \overbrace{\min(F_i(x)+F_i'(x)d, G_i(x)+G_i'(x)d)}^{\text{one is } 0, \text{ so } \min \leqslant 0}$$

$$+ \sum_{i\in\mathcal{E}^{-}(x)} \overbrace{H_i(x)}^{<0} \overbrace{\min(F_i(x)+F_i'(x)d, G_i(x)+G_i'(x)d)}^{\text{both are } \geqslant 0, \text{ so } \min \geqslant 0}$$

$$= -2\theta(x) - \dots - \dots \leqslant -2\theta(x) \quad (< 0)$$

PNM computations

$$\theta'(x;d) = \sum_{i\in\mathcal{F}(x)} F_i(x)F_i'(x)d + \sum_{i\in\mathcal{G}(x)} G_i(x)G_i'(x)d + \sum_{i\in\mathcal{E}(x)} H_i(x)\min(F_i'(x)d, G_i'(x)d)$$

$$[d \text{ s.t.} \ldots] = -2\theta(x) \quad [\text{smooth}] \qquad [\downarrow \text{nonsmooth} \downarrow]$$

$$+ \sum_{i\in\mathcal{E}^{0+}(x)} \overbrace{H_i(x)}^{\geqslant 0} \overbrace{\min(F_i(x)+F_i'(x)d, G_i(x)+G_i'(x)d)}^{\text{one is } 0, \text{ so min } \leqslant 0}$$

$$+ \sum_{i\in\mathcal{E}^{-}(x)} \overbrace{H_i(x)}^{<0} \overbrace{\min(F_i(x)+F_i'(x)d, G_i(x)+G_i'(x)d)}^{\text{both are } \geqslant 0, \text{ so min } \geqslant 0}$$

$$= -2\theta(x) - \cdots - \cdots \leqslant -2\theta(x) \quad (<0)$$

PNM computations

$$\theta'(x;d) = \sum_{i\in\mathcal{F}(x)} F_i(x)F_i'(x)d + \sum_{i\in\mathcal{G}(x)} G_i(x)G_i'(x)d$$

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$$= -2\theta(x) - \dots - \dots \leqslant -2\theta(x) \quad (<0)$$

PNM computations

$$\theta'(x; d) = \sum_{i \in \mathcal{F}(x)} F_i(x) F_i'(x) d + \sum_{i \in \mathcal{G}(x)} G_i(x) G_i'(x) d$$

$$+ \sum_{i \in \mathcal{E}(x)} H_i(x) \min(F_i'(x) d, G_i'(x) d)$$

$$[d \text{ s.t.} \dots] = \; -2\theta(x) \quad [\text{smooth}] \qquad [\downarrow \text{nonsmooth} \downarrow]$$

$$+ \sum_{i \in \mathcal{E}^{0+}(x)} \overbrace{H_i(x)}^{\geqslant 0} \overbrace{\min(F_i(x) + F_i'(x) d, G_i(x) + G_i'(x) d)}^{\text{one is } 0, \text{ so min } \leqslant 0}$$

$$+ \sum_{i \in \mathcal{E}^{-}(x)} \overbrace{H_i(x)}^{< 0} \overbrace{\min(F_i(x) + F_i'(x) d, G_i(x) + G_i'(x) d)}^{\text{both are } \geqslant 0, \text{ so min } \geqslant 0}$$

$$= -2\theta(x) - \dots - \dots \leqslant -2\theta(x) \quad (< 0)$$

Explicit computations

Let $\Gamma_+ = \mathrm{Diag}(\gamma_+), \Gamma_- = \mathrm{Diag}(\gamma_-)$

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$$\theta'(x; -g(\gamma_+, \gamma_-)) = -||g(\gamma_+, \gamma_-)||^2$$
$$+ H^{\mathsf{T}}_{\mathcal{E}^{0+}(x)} \underbrace{[\min(-F'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-), -G'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-)) + \Gamma_+ F'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-) + \overline{\Gamma}_+ G'_{\mathcal{E}^{0+}(x)} g(\gamma_+, \gamma_-)]}$$
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$$\min(-a, -b) + \gamma a + \overline{\gamma} b = \begin{cases} \overline{\gamma}(b-a) \leqslant 0 & \text{if } a \geqslant b \\ \gamma(a-b) \leqslant 0 & \text{if } a \leqslant b \end{cases}.$$

Computation of the weights

Goal: get a nonzero g

$$\max_{\gamma_+ \in [0,1]^{\mathcal{E}^{0+}(x)}} \min_{\gamma_- \in [0,1]^{\mathcal{E}^{-}(x)}} ||g(\gamma_+, \gamma_-)||^2/2$$

where $g(\gamma_+, \gamma_-) = g_0 + M_+\gamma_+ + M_-\gamma_-$.

min: $\gamma_-(\gamma_+)$ to descend; max: distance (convex) on hypercube: $\rightsquigarrow \{0,1\}^{\mathcal{E}^{0+}(x)}$ (partitions); inclusion of polytopes: vertices suffice.

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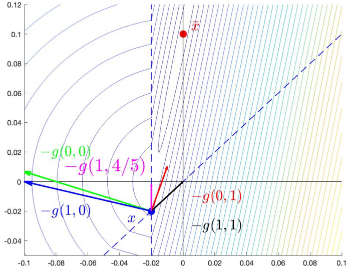
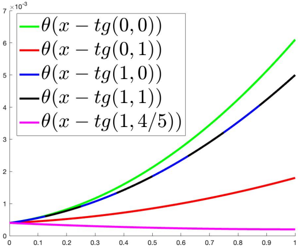
Example of weights computation – 1





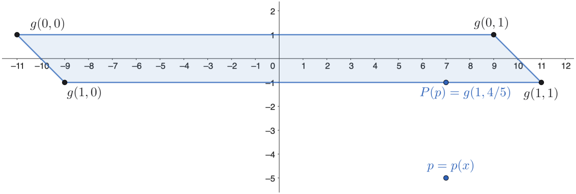
$$M = \begin{pmatrix} 1/2 & 1/2 \\ -5 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ -1/10 \end{pmatrix}, \quad x = \begin{pmatrix} -1/50 \\ -1/50 \end{pmatrix}$$

Here, $\mathcal{F}(x) = \varnothing = \mathcal{G}(x)$, $\mathcal{E}(x) = \{1, 2\}$. After computations. . .

Example of weights computation – 2

[up to factor $1/200$] The point to project is $(7; -5)(= g_1(x))$.

The zonotope to project on is $\begin{bmatrix} 1 & 10 \\ -1 & 0 \end{bmatrix} \times [-1, +1]^2$



The projection is $(7; -1) = 4/5 \times (11; -1) + 1/5 \times (-9; -1)$, and also $g(1, 4/5) = 4/5 g(1, 1) + 1/5 g(1, 0)$

Some precisions for LCPs (1)

$$0 \leqslant F(x) = x \perp Mx + q = G(x) \geqslant 0$$
$$A = \mathcal{F}(x), \quad E = \mathcal{E}(x), \quad I = \mathcal{G}(x)$$

$$\mathcal{C}_{\text{ND}} \begin{cases} M_{I,I} \text{ nonsingular and} \\ M_{E,E} - M_{E,I} M_{I,I}^{-1} M_{I,E} \in \boldsymbol{ND} \end{cases}$$

Weighted framework for LCPs with $\mathcal{C}_{\text{ND}}$

The following properties are equivalent:

1) x is a solution,
2) for any (γ_+, γ_-), $g(\gamma_+, \gamma_-) = 0$,
3) for some $(\gamma_+, \gamma_-) \in \{0,1\}^{\mathcal{E}(x)}$, $g(\gamma_+, \gamma_-) = 0$.

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Algorithm properties

Sufficient decrease

1) $d_k(\lambda)/||d_k(\lambda)|| \underset{\lambda \to +\infty}{\to} -S_k^{-1}g_k/||S_k^{-1}g_k||$
2) for λ large enough, a descent formula holds

Convergence

Let (x_k, λ_k, S_k) be a sequence generated by algorithm 1.

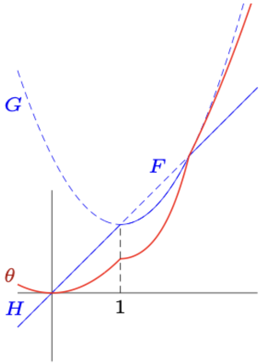
1) The sequence $(\theta(x_k))_k$ decreases thus converges.
2) If $(F'(x_k), G'(x_k), \lambda_k S_k)_{k \in \mathcal{K}}$ for a subsequence $\mathcal{K}$ is bounded, then $g_k \to 0$.

In particular, "good behavior" of algorithm is assumed: $\lambda_k \nrightarrow +\infty$.

"Concave kinks" – difficult (bad) limit points.

Consider a simple example with
$n = 1$, $F(x) = x$,
$G(x) = 1 + (x - 1)^2$, $x_0 = 3/2$.

For $x \in (1, 2)$, $F(x) \neq G(x)$.



Counter-example

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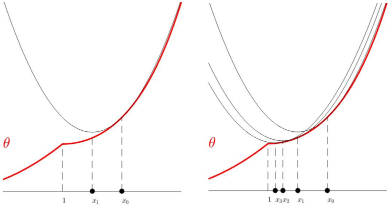
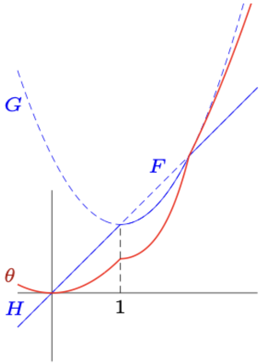


First iterates, convergence to $x = 1$. The black curves are the quadratic models ψ_x.



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